

Characterization of Reward Functions in Networks with Costs

Antoni Meseguer** and Raquel Ferreras
Avinguda Tibidabo, 39-43, 08035 Barcelona

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Abstract

In this paper we study network structures in which the possibilities for cooperation are restricted and can not be described by a cooperative game. The benefits of a group of players depend on how these players are internally connected . One way to represent this type of situations is the so-called reward function, which represents the profits obtainable by the total coalition if links can be used to coordinate agents' actions. The starting point of this paper is the work of Vilaseca *et al.* [10] where they characterized the reward function. We concentrate on those situations where there exist costs for establishing communication links. Given a reward function and a costs function, our aim is to analyze under what conditions it is possible to associate a cooperative game to it. We characterize the reward function in networks structures with costs for establishing links by means of two conditions, component permanence and component additivity. Finally, an economic application is developed to illustrate the main theoretical result.

Keywords: cooperative game; network; reward function; costs function.

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**Corresponding author. Email: ameseguer@uoc.edu

1 Introduction

In this paper, we will focus on a particular field of the scientific literature that studies how networks integrate into cooperative games. The aim of this work is to study how exogenous rules of sharing determine the benefits of cooperation between agents. Additionally, we will consider those economic situations in which the communication possibilities among a group of agents are restricted and where the formation of connections between agents has a cost.

Most part of communication takes place through networks, systems of decentralized and bilateral relations between participants. In recent years there has been a growing attention for the theoretical models which remark the importance of the economic and social networks. These models help us to understand how networks affect to economic outcomes and analyze how networks are formed.

Myerson [6] was the precursor of the study of communication situations. In his work he associated a situation where the possibilities for communication are restricted to the so-called network-restricted game, in which he incorporates the benefits of cooperation and the communication restrictions. Later, Borm *et al.* [3] contributed with a new approximation that was based on the possible connections that permit communication among agents, and not from the point of view of players. The modelization was made through the idea of link game, that reflects for each subset of existent connections, the value that will obtain the total coalition if only these communication channels are feasible to players.

The previous works assume that the economic possibilities of agents can be described via a cooperative game. From now on, we will omit this supposition. In this sense, it is known that there are situations where the profits obtainable by a set of players do not only depend on the connected agents, but furthermore they depend on how these players are internally connected. These situations are represented via the so-called reward function, introduced by Jackson and Wolinsky [5], which measures the benefits obtainable by the grand coalition in the network. The main difference with respect to the link game is that the link game is only defined for a concrete connection of the network, while the reward function is defined over all set of possible links.

But there is a step behind where there are costs for forming links. The formation of communication networks when there exists costs has been studied in different works as Bala and Goyal [2], Goyal [4], Jackson and Wolinsky [5], Slikker and van den Nouweland [8] and Watts [11].

While the first three works study the formation of networks in parametric models of information transmission, Slikker and van den Nouweland [8]

analyze the influence of the existence of costs for establishing links in formed communication networks. For a wide summary of the literature we refer to Slikker and van den Nouweland [9].

The starting point of our work is the modelling of Jackson and Wolinsky [5] through the reward function, but taking into account the discussion in Slikker and van den Nouweland [8]. Given a cooperative game it is always possible to associate to it a reward function, however the inverse implication is not always true. In this sense, Vilaseca *et al.* [10] characterized the reward function, proving that a reward function has a unique cooperative game associated if and only if the reward function satisfies two conditions: the component permanence and the component additivity.

In this paper, we will follow the research line started by Vilaseca *et al.* [10], with the difference that we will assume the existence of costs for establishing communication links between every two players.

The structure of the paper is as follows. In Sec. 2 we recall some basic game theoretic notions and we provide some necessary definitions and concepts about the Theory of Networks. In Sec. 3 we give the main result of the paper, the characterization of the reward function when costs are taken into account in the establishment of links. In Sec. 4 we show an economic application of the theoretical result, relied on the macroeconomic vision of the weight of some countries in the distribution of votes that proposes the European Constitution based on demographic criteria. And finally, in Sec. 5 we conclude with some final remarks.

2 Notation and preliminaries

2.1 Cooperative games and solution concepts

A cooperative game is a pair (N, v) where $N = \{1, 2, \dots, n\}$ represents the set of players and v is the characteristic function describing the gains of cooperation, $v(S) \in \mathbb{R}$, for each coalition $S \subseteq N$. It is always assumed that $v(\emptyset) = 0$.

These games are also referred to as cooperative games with transferable utility or TU-games. The concept of transferable utility refers to the fact that the proceeds from cooperation are transferable between the players.

In this paper we will be specially interested in 0-normalized non-negative games, which is to say cooperative games where $v(\{i\}) = 0$ for all $\{i\} \in N$ and $v(S) \geq 0$ for all $S \subseteq N$. To simplify notation, from now on we will write i instead of $\{i\}$ to refer to player i .

Players in a cooperative game are not just interested in profits obtainable

in a coalition, furthermore they can be interested in what they individually get from cooperation. In this sense, Game Theory provides single-valued solution concepts, such as the well-known Shapley value [7] which distributes the total value among players taking into account the concept of marginal contribution. An allocation or payoff vector is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^n$ that assigns to each player $i \in N$ the payoff or profit x_i that can be obtained if he cooperates with the other players.

Given a cooperative game (N, v) the Shapley value $\phi(v) \in \mathbb{R}^n$ is defined as:

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} (v(S \cup i) - v(S)), \quad i = 1, 2, \dots, n.$$

2.2 Networks

In this section, we will focus on describing communication situations. The main difference with the previous section is that there we assumed that all coalitions of players could be formed, but in certain situations this is not the case.

Given a set of players $N = \{1, 2, \dots, n\}$, in order to be able to coordinate their actions they have to be able to communicate. A communication network that describes the bilateral channels of communication between players can be represented by a graph (N, L) where $N = \{1, 2, \dots, n\}$ is the set of players that we will situate in the vertices or nodes of the graph and where these players are connected by a set of arcs or links $L \subseteq L^N = \{\{i, j\} | i, j \in N, i \neq j\}$. For instance, if $N = \{1, 2, 3\}$ and $L = \{\{1, 2\}, \{2, 3\}\}$ then it indicates that there is a link between individuals 1 and 2, a link between individuals 2 and 3, but there is no link between individuals 1 and 3.

The link $\{i, j\}$ indicates that players i and j are directly connected in the graph and can cooperate with each other without requiring the intermediation of other players. We say that i and j are connected in the network if there is a path that joins them, that is to say, if there is a sequence of players (i_1, i_2, \dots, i_t) such that $i_1 = i$, $i_t = j$ and $\{i_k, i_{k+1}\} \in L$ for all $k \in \{1, 2, \dots, t-1\}$. If two players are connected, but are not directly connected, we say that they are indirectly connected in the network.

A network (N, L) is complete if all pairs of players are directly connected in the network, i.e., $L = L^N$. Given a coalition $S \subseteq N$ where $|S| = 2, 3, \dots, n$, we denote with L_S the structures of links that result in complete networks. $L_S = \{\{i, j\} | i, j \in S\}$ is a structure of links where the players of S are all directly connected and the rest of the players are not connected.

Note that if there are players who are not connected in a network, this

means that they cannot communicate and, therefore, it rises to the notion of cooperation components, i.e., a partition of the set of players N . We say that i and j are in the same component C if they are connected, either directly or indirectly. The set of components of the network (N, L) will be denoted as N/L . To any component C we can associate a coalition that we will call S_C such that $S_C = \{i \in N/i \in C\}$.

To coordinate actions between players of a component C , only players of this component are relevant. Hence the network associated to the component C is denoted $L(C) = \{\{i, j\} \in L | i, j \in C\}$, i.e., $L(C)$ represents the network where there are only the links of C and where the individual players $k \in N \setminus C$ are not connected with anyone. For example, if we consider the network (N, L) where $N = \{1, 2, 3, 4, 5\}$ and $L = \{\{1, 2\}, \{1, 3\}, \{4, 5\}\}$, then the set of components of (N, L) is $N/L = \{\{1, 2, 3\}, \{4, 5\}\}$ and, for example, the network associated to component $C = \{1, 2, 3\}$ is $L(\{1, 2, 3\}) = \{\{1, 2\}, \{1, 3\}\}$.

3 Characterization of reward function in networks structures with costs

3.1 Definition of reward function and costs function

There are some situations in which the economic possibilities of players cannot be described by a cooperative game. Borm *et al* [3] used the so-called link game to represent such situations, while Jackson and Wolinsky [5] defined the so-called reward function which considers the internal structure of links between players.

We will concentrate in Jackson and Wolinsky approximation via the reward function. The reward function is a function that assigns a real value $r(L)$ to each set of links $L \subseteq L^N$, that represents the profits obtainable by the grand coalition in network (N, L) if the connections of L can be used to coordinate players' actions.

Additionally it is possible to consider that forming communication links among players is not costless. Slikker and van den Nouweland [8] incorporate costs for forming links and they assume that all links result in the same fixed cost. We will work with a more general model of costs function where the cost associated to each link can be different.

Definition 3.1 *For all set of links $L \subseteq L^N$, a costs function associated to each possible connections is defined as a function:*

$$f : 2^{L^N} \rightarrow \mathbb{R} \quad \text{where } f(\emptyset) = 0. \quad (1)$$

For all cooperative game (N, v) and all costs function $f(L)$ a reward function r is associated in the following way:

$$r(L) = \sum_{C \in N/L} v(C) - f(L) \quad \text{for all } L \subseteq L^N. \quad (2)$$

That is to say, the value for a set of links L is defined as the profits that would obtain the grand coalition if connections of L can be used to coordinate players' actions.

For this reward function and costs function two properties can be defined:

Definition 3.2 *Given a set of players N and a function z defined on subsets of L^N , the function z is component permanent if it holds that:*

$$z(L_1) = z(L_2) \quad (3)$$

for all $L_1, L_2 \subseteq L^N$ such that $N/L_1 = N/L_2$.

Definition 3.3 *Given a set of players N and a function z defined on subsets of L^N , the function z is component additive if it holds that:*

$$z(L) = \sum_{C \in N/L} z(L(C)) \quad \text{for all } L \subseteq L^N. \quad (4)$$

3.2 Characterization of reward function

In this subsection we will focus on characterizing the reward function. To be precise we will prove that given two functions, r and f , defined for all $L \subseteq L^N$, where $f(\emptyset) = 0$, component permanence and component additivity of $r + f$ are necessary and sufficient conditions for a unique 0-normalized associated cooperative game to exist that has as reward function exactly this function.

Next we will check that if the reward function and the costs function don't generate externalities, then the value of the reward function is null for the case where there is no connection between nodes.

Lemma 3.1 *Given a set of players $N = \{1, 2, \dots, n\}$ such that $|N| \geq 2$, a costs function $f : 2^{L^N} \rightarrow \mathbb{R}$ where $f(\emptyset) = 0$ and a reward function $r : 2^{L^N} \rightarrow \mathbb{R}$ such that $r + f$ is component additive, then $r(\emptyset) = 0$.*

Proof: Without loss of generality we define $r' = r + f$. Due to component additivity of r' , then $r'(L) = \sum_{C \in N/L} r'(L(C))$ for any $L \subset L^N$ and so,

$$r'(\emptyset) = \sum_{i \in N} r'(L(i)) = \sum_{i \in N} r'(\emptyset) \Rightarrow r(\emptyset) + f(\emptyset) = n \cdot (r(\emptyset) + f(\emptyset)).$$

Since by definition $f(\emptyset) = 0$, then $r(\emptyset) = n \cdot r(\emptyset) \Rightarrow r(\emptyset) = 0$.

□

The following theorem is the main result of this paper and characterizes reward function in those situations where there are costs for establishing links.

Theorem 3.1 *Given a 0-normalized cooperative game (N, v) , a costs function f (1) and the reward function r of the game (2), the function $r + f$ satisfies component permanence (3) and component additivity (4).*

Conversely, given a costs function f and a function $r : 2^{L^N} \rightarrow \mathbb{R}$ such that $r + f$ satisfies (3) and (4), there exists a unique 0-normalized game (N, v) such that its reward function is r .

Proof:

\Rightarrow) We have to prove that given a 0-normalized cooperative game (N, v) , a costs function f and a reward function r , then the function $r + f$ satisfies (3) and (4).

(1) Given $L_1, L_2 \subseteq L^N$ such that $N/L_1 = N/L_2$ and given a cooperative game v and a costs function f , if we associate the reward function defined in (2), then it satisfies that:

$$\begin{aligned} r(L_1) &= \sum_{C \in N/L_1} v(C) - f(L_1) \Rightarrow r(L_1) + f(L_1) = \sum_{C \in N/L_1} v(C), \\ r(L_2) &= \sum_{C \in N/L_2} v(C) - f(L_2) \Rightarrow r(L_2) + f(L_2) = \sum_{C \in N/L_2} v(C), \end{aligned}$$

so,

$$r(L_1) + f(L_1) = \sum_{C \in N/L_1} v(C) = \sum_{C \in N/L_2} v(C) = r(L_2) + f(L_2).$$

- (2) We define $r' = r + f$. So, now we have to prove that r' satisfies component additivity.

Without loss of generality, we consider the following components structure $N/L = \{C_1, C_2, \dots, C_k\}$. Then, according to expression (2) of the reward function associated to v , we have:

$$r'(L) = \sum_{C \in N/L} v(C) = v(C_1) + v(C_2) + \dots + v(C_k). \quad (5)$$

From expression (2) and 0-normalization of v , the value for a structure of links associated to a unique component C_i for all $i \in \{1, 2, \dots, k\}$ is:

$$r'(L(C_i)) = \sum_{C \in N/L(C_i)} v(C) = v(C_i) + \sum_{j \in N \setminus C_i} v(j) = v(C_i).$$

Then, substituting in (5) we obtain:

$$\begin{aligned} r'(L) &= v(C_1) + v(C_2) + \dots + v(C_k) = \\ &= r'(L(C_1)) + r'(L(C_2)) + \dots + r'(L(C_k)) = \\ &= \sum_{i=1}^k r'(L(C_i)) = \sum_{C \in N/L} r'(L(C)). \end{aligned}$$

Hence, we can affirm that r' is component additive.

- \Leftarrow) We have to prove that given a costs function f and a function $r : 2^{L^N} \rightarrow \mathbb{R}$ such that $r + f$ satisfies (3) and (4), then there exists a unique 0-normalized cooperative game v such that its reward function is r .

We will divide the proof in two steps. First we will prove the existence of the game and secondly the uniqueness.

To show the existence we will prove that there exists a 0-normalized cooperative game v that satisfies the system of equations described by (2).

Consider the following cooperative game where $v(i) = 0$ for all $i \in N$ and $v(S) = r(L_S) + f(L_S)$ for all $S \subseteq N$, $|S| \geq 2$.

Consider $L \subseteq L^N$ such that L has a unique component that is not a singleton and that is formed by players in coalition S . Then by 0-normalization of v , and component permanence of $r + f$, it holds that:

$$r(L) + f(L) = r(L_S) + f(L_S) = v(S) = v(S) + \sum_{j \in N \setminus S} v(j).$$

Then, for such types of networks, v satisfies (2).

Similarly, consider now $L \subseteq L^N$ such that L has more than one component that are not singletons, and let these components be formed by coalitions S_1, S_2, \dots, S_k . Then by 0-normalization of v , and component permanence and additivity of $r + f$, it holds that

$$\begin{aligned} r(L) + f(L) &= \sum_{i=1}^k r(L_{S_i}) + \sum_{i=1}^k f(L_{S_i}) = \\ &= \sum_{i=1}^k v(S_i) = \sum_{i=1}^k v(S_i) + \sum_{j \in N \setminus \{S_1 \cup S_2 \cup \dots \cup S_k\}} v(j). \end{aligned}$$

That is to say, v also satisfies condition (2).

Last, the case $L = \{\emptyset\}$ follows directly from Lemma 3.1.

Finally, to show uniqueness, suppose that there exist two 0-normalized cooperative games v_1 and v_2 which are solution of the equations system described by (2). Then there exists a coalition $S \subset N$, $|S| \geq 2$, such that $v_1(S) \neq v_2(S)$. Then by (2) and 0-normalization of v_1 and v_2 , it holds that:

$$\begin{aligned} r(L(S)) - f(L(S)) &= v_1(S) + \sum_{j \in N \setminus S} v_1(j) = v_1(S), \\ r(L(S)) - f(L(S)) &= v_2(S) + \sum_{j \in N \setminus S} v_2(j) = v_2(S). \end{aligned}$$

Equations which lead to a contradiction, because we had supposed that $v_1(S) \neq v_2(S)$.

□

The following corollary tells us that the 0-normalization is a necessary and sufficient condition for the function $r + f$ to be component additive.

Corollary 3.1 *Let be (N, v) a cooperative game, $f : 2^{L^N} \rightarrow \mathbb{R}$ a costs function where $f(\emptyset) = 0$ and $r : 2^{L^N} \rightarrow \mathbb{R}$ its reward function, then (N, v) is 0-normalized if and only if $r + f$ satisfies component additivity.*

Proof:

\Rightarrow) According to the proof of Theorem 3.1, if a game v is 0-normalized then the function $r + f$ is component additive.

\Leftarrow) If v is not 0-normalized, then $\sum_{j \in N} v(j) \neq 0 \Rightarrow r(\emptyset) = \sum_{j \in N} v(j) - f(\emptyset) \neq 0$, applying Lemma 3.1 we can affirm that $r + f$ is not component additive. This leads to a contradiction and proves the corollary.

□

The following proposition shows that given a costs function and the reward function for complete networks, then it is possible to obtain the value of the reward function for the rest of links $L \subseteq L^N$.

The interest of the result lies in that the complexity of the definition of the reward function r is reduced, due to the fact that just associating its value for complete networks, it remains determined.

Proposition 3.1 *Given a set of players $N = \{1, 2, \dots, n\}$ such that $|N| \geq 2$, a costs function $f : 2^{L^N} \rightarrow \mathbb{R}$ and a reward function $r : 2^{L^N} \rightarrow \mathbb{R}$ such that $r + f$ satisfies conditions (3) and (4), then r is fully determined by f and $r(L_{S_C})$:*

$$r(L) = \sum_{C \in N/L} r(L_{S_C}) + \sum_{C \in N/L} f(L_{S_C}) - f(L) \quad (6)$$

for all $L \subseteq L^N$.

Proof: Without loss of generality, consider $L \subseteq L^N$ such that $N/L = \{C_1, C_2, \dots, C_k\}$ and consider $L^* \subseteq L^N$ such that $L^* = \{\{r, j\} \in L/r, j \in C_i\}$ for all $i = \{1, 2, \dots, k\}$. Observe that $N/L^* = N/L = \{C_1, C_2, \dots, C_k\}$.

Hence, by component permanence (3) of $r + f$:

$$r(L) + f(L) = r(L^*) + f(L^*) \Rightarrow r(L) = r(L^*) + f(L^*) - f(L) \quad (7)$$

for all $L, L^* \subseteq L^N$.

By component additivity (4) of $r + f$:

$$\begin{aligned} r(L^*) + f(L^*) &= \sum_{i=1}^k [r(L_{S_{C_i}}) + f(L_{S_{C_i}})] \Rightarrow \\ \Rightarrow r(L^*) &= \sum_{i=1}^k r(L_{S_{C_i}}) + \sum_{i=1}^k f(L_{S_{C_i}}) - f(L^*). \end{aligned}$$

Replacing the previous expression in (7) we obtain:

$$\begin{aligned} r(L) &= \sum_{i=1}^k r(L_{S_{C_i}}) + \sum_{i=1}^k f(L_{S_{C_i}}) - f(L^*) + f(L^*) - f(L) \Rightarrow \\ \Rightarrow r(L) &= \sum_{C \in N/L} r(L_{S_C}) + \sum_{C \in N/L} f(L_{S_C}) - f(L). \end{aligned}$$

□

Economically the analysis of this expression tells that if all functions are positive, then higher connections being formed between different agents imply higher costs and so, the reward function takes a higher value for those networks with less links, due to the fact that in this case there are less costs and more benefits.

From the proof of Proposition 3.1 it can be deduced that the equations system described previously in (6) is an indeterminate compatible system with $2^n - n - 1$ free parameters, that is to say, it has so many degrees of freedom as number of coalitions can be formed.

It is easy to check that from Proposition 3.1 it is also possible to obtain the value of the costs function if we have the reward function and the values of the costs function for complete networks. The expression in that case is:

$$f(L) = \sum_{C \in N/L} f(L_{S_C}) + \sum_{C \in N/L} r(L_{S_C}) - r(L) \quad (8)$$

for all $L \subseteq L^N$.

3.3 Particular case with costs proportional to the number of links

Next we consider one of the most habitual cases, the one where the costs function is proportional to the number of links:

$$f(L) = \alpha|L| \quad \text{for all } L \subseteq L^N \quad \text{where } \alpha > 0. \quad (9)$$

The goal is to analyze what type of structure needs to have the reward function when the associated costs function is the one described previously in (9) and, taking into account that $r + f$ accomplishes conditions (3) and (4).

From Proposition 3.1 it is possible to express $r(L)$ as a function of parameter α , the number of links and the values of the reward function for

complete networks. The value of the reward function for any connection, considering proportional costs, is:

$$r(L) = \sum_{C \in N/L} r(L_{SC}) + \alpha \left(\sum_{C \in N/L} |L_{SC}| - |L| \right) \quad (10)$$

for all $L \subseteq L^N$.

The interpretation of expression (10) from the economic point of view and considering that the costs function is increasing ($\alpha > 0$) is that higher number of links being formed between different agents of the network leads to more costs and, so, to a decreasing network value.

4 Economic application: A macroeconomic approach to the countries votes allocation in European Constitution

In this section we compare the present assigned votes to different countries in European Constitution according basically to demographic criteria with the votes that would be assigned if other kind of criteria were followed, as for instance the external commercial relationships of considered economies.

After the Lisboa Treaty, European Council decisions are taken according to a complex weighting system mostly depending on the demographic weight of each of the 24 European Union Member States. From the total 345 votes attributable, countries with more percentage of votes are Germany, United Kingdom, France and Italy with 29 votes, followed by Spain and Poland with 27 votes each one.

For reasons of data availability and to simplify the analysis, we will work only with the following four countries: Germany, France, Italy and Spain.

The economic application will be modeled via a network structure, where the different countries will be located in the nodes and the links will indicate the existence of external trade between states. Graphically, the network situation that we will study is represented in Figure 1.

In this sense we will consider, as indicator of international trade of goods an aggregated economic magnitude, the exports flow detailed by countries partners. Data referent to 2005, evaluated in millions of dollars, have been obtained from OECD (<http://www.oecd.org>). With the aim of homogenizing the difference of living standards we will correct the present exchange rate by the purchasing power parity.

These data will permit us to obtain the value of the reward function for each network structure, that will correspond with the increment generated

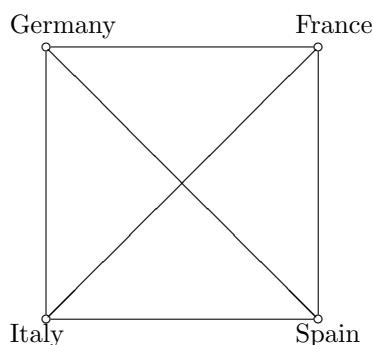


Figure 1: Connections structure

in the Gross Domestic Product (GDP), aggregated and with no exports, by the fact of considering the external relations in terms of exports between different countries (Step 1). Additionally, we consider the incorporation of costs in the formation of links with the objective of satisfying conditions of Theorem 3.1. Hence, we will be able to construct an associated cooperative game (Step 2), that will permit us to obtain the weights of each country in terms of macroeconomic magnitudes (Step 3).

Step 1: As we aforementioned we define the values of networks structures in terms of foreign relations. Specifically, the value of a particular network will be given by the net productivity, in terms of value market of produced final goods, generated by different countries when there are no trade exchanges (GDP minus exports) plus the generation of value obtained through foreign trade relations (exports) among connected countries in that network.

These values will permit us to construct the reward function. The reward function indicates the increment in the value generation produced when we pass from a situation where there is no trade movement to a situation where foreign relations are taking into account.

Table 1 shows the Gross Domestic Product (GDP) of each one of the states, as well as global exports, in millions euros. Table 2 shows exports values, in millions euros, between different considered countries. The computation of the reward function is let to the reader, due to the fact that the number of combinations of connections structures is very high. We will normalize with respect to the case of closed economies, $L = \{\emptyset\}$, in order to achieve the values of each one of possible networks. Without loss of generality, from now on we call Germany as player 1, France as player 2, Italy as player 3 and Spain as player 4.

	Germany	France	Italy	Spain
GDP	2122,1	1562,6	1232,7	740,1
Exports	802,1	371,7	248,4	151,1

Table 1: GDP and global exports (millions of euros)

exporters \ importers	Germany	France	Italy	Spain
Germany		56,37	42,79	19,56
France	90,12		41,27	34,14
Italy	57,86	34,88		13,78
Spain	37,80	33,99	20,75	

Table 2: Exports between countries (millions of euros)

For example, the value of reward function for links structure $L = \{\{1, 2\}, \{3, 4\}\}$, is given by:

$$\begin{aligned}
r(\{\{1, 2\}, \{3, 4\}\}) &= \\
&= (2122, 1 + 1562, 6 + 1232, 7 + 740, 1) - (802, 1 + 371, 7 + 248, 4 + 151, 1) + \\
&+ (56, 37 + 90, 11) + (13, 78 + 20, 75) = 4265, 21
\end{aligned}$$

it means, the economic interpretation of this worth is the added value to GDP minus exports generated by all four countries, that is obtained by considering only the exports between Germany and France and between Italy and Spain.

The reader can easily check that [10] does not apply, since for example $r(\{\{1, 2\}, \{1, 3\}\}) \neq r(\{\{1, 3\}, \{2, 3\}\})$, that is to say, condition of component permanence is not satisfied.

Step 2: In the description of the model, we assume that formation of communication links between any two players results in a non-negative exogenous cost. To isolate the effect of costs in structures that are formed, we assume equal costs for all possible communication links.

The costs function of a network structure consists of a fix and a variable part. Fixed costs are given by exports between countries which are not connected in complete networks associated to each one of the components, and the variable costs are proportional to the number of links of complete networks associated to the components. The expression of costs function, where X_L represents exports between connected countries of network L , is given by the following equation:

$$f(L) = \sum_{C \in N \setminus L} X_{L_{S_C} \setminus L(C)} + \alpha \sum_{C \in N \setminus L} |L_{S_C}| \quad \text{where } \alpha \geq 0, \quad (11)$$

and where by definition $f(\emptyset) = 0$.

So for instance, the value of the costs function for the connections structure $L = \{\{1, 2\}, \{1, 3\}\}$ is given by:

$$\begin{aligned} f(\{\{1, 2\}, \{1, 3\}\}) &= X_{\{\{1,2\},\{1,3\},\{2,3\}\} \setminus \{\{1,2\},\{1,3\}\}} + |\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}| \alpha \\ &= (41, 27 + 34, 88) + 3\alpha \end{aligned}$$

By the definition of the reward function and the costs function, it is easy to check that as much component permanence condition as component additivity condition of function $r + f$ are satisfied, therefore, following Theorem 3.1 we can associate a cooperative game to the reward function.

To obtain the cooperative game associated to the reward function we have to solve the equations system described in expression (2). As it is deduced from proof of Theorem 3.1, the solution of the system is quasi immediate.

Given the costs function described in (11), the cooperative game (N, v) , where $N = \{1, 2, 3, 4\}$ and v is expressed in millions of euros, associated to the previous reward function has the following characteristic function:

$$\begin{array}{lll} v(\emptyset) = 0 & v(\{1, 2\}) = 146.49 + \alpha & v(\{1, 2, 3\}) = 304.5 + 3\alpha \\ v(\{1\}) = 0 & v(\{1, 3\}) = 100.66 + \alpha & v(\{1, 2, 4\}) = 271.99 + 3\alpha \\ v(\{2\}) = 0 & v(\{1, 4\}) = 57.36 + \alpha & v(\{1, 3, 4\}) = 192.55 + 3\alpha \\ v(\{3\}) = 0 & v(\{2, 3\}) = 76.16 + \alpha & v(\{2, 3, 4\}) = 178.83 + 3\alpha \\ v(\{4\}) = 0 & v(\{2, 4\}) = 68.14 + \alpha & v(\{1, 2, 3, 4\}) = 483.33 + 6\alpha \\ & v(\{3, 4\}) = 34.53 + \alpha & \end{array}$$

Precisely, the value of the game for any coalition S coincides with the value of the reward function for the component associated to this coalitions plus a proportional cost $\alpha \geq 0$ to the number of links of the complete network associated to the component.

An implication of the increasing costs function is that higher number of foreign exchanges between countries leads to more benefits.

Step 3: Once the cooperative game has been obtained, the following step consists in using an external distribution rule to determine the payments to different countries.

The individual allocation that we will use is the Shapley value, which is an extension of the Myerson value used for situations where there are

communication restrictions (Aumann and Myerson [1]). According to this concept solution, the share-out of the total profits is:

$$\phi(v) = (150.685 + 1.5\alpha, 143.825 + 1.5\alpha, 104.105 + 1.5\alpha, 84.715 + 1.5\alpha).$$

If we take into account the economic criteria of foreign openness, Germany is the state which had to receive a higher votes percentage in European Constitution, followed by France, Italy and Spain. If we compare these percentages with the ones assigned following demographic criteria we observe that they differ. On the one hand the sharing that follows the demographic criteria assigns to Germany, France and Italy the same number of votes, but on the other hand Spain is the one which gets less percentage of votes. Note that the costs effect over different countries is totally marginal, due to the fact that by the own definition of the costs function those costs are included additively in the characteristic function.

5 Conclusions

In this article, we have studied situations where communication between players can be restricted and where the formation of these communication links has a cost. To do it, we consider the reward function, introduced by Slikker and van den Nouweland [9], as the analysis tool. The reward function is a function that assigns to each connections structure formed by players and links, a value that indicates the profits obtainable by the grand coalition in this situation.

Given a set of players and a cooperative game, it is always possible to associate a reward function to the game. However, it is not always possible to associate a cooperative game to a reward function. Following Vilaseca *et al.* [10], we characterize the reward function for these situations where the establishment of relations among agents has a cost.

So given a reward function and a costs function, we show which conditions must be satisfied to have a unique associated cooperative game. These conditions are the component permanence and the component additivity of the function that is the sum of the reward function and the costs function. This is the main result of the paper.

Futhermore, as a consequence of the main result we prove that given a cooperative game, a costs function and its reward function, then the game is 0-normalized if and only if the function that results from the sum of r and f is component additive.

Once we have the theoretical tool, we show an empirical application of the result. The economic example that we present compares the actual distribution of votes that assigns the European Constitution to different countries according to demographic criteria with the distribution that would take place if we would consider macroeconomic criteria such as the foreign commercial relationships.

According to this macroeconomic approach of external commercial relations, the state that should receive more percentage of votes is Germany, followed by France, Italy and Spain, allocation which differs from the one imputed following demographic criteria.

With respect to future research lines, from the theoretical point of view one of the main objectives is to analyze what conditions must satisfy the reward function in order to have a unique cooperative game not necessarily 0-normalized. From the empirical side, we will apply this technique to the study of other kind of networks, such as logistic or telecommunication networks.

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