Máster en Ingeniería Computacional y Matemática

Trabajo Final de Máster

Malliavin calculus applied to option pricing theory

Marc Lagunas Merino

Director del Trabajo Final de Máster:
Dr. Josep Vives Santa-Eulalia

Profesor responsable de la Asignatura:
Dr. Juan Alberto Rodríguez Velázquez

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FICHA DEL TRABAJO FINAL

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Nombre del autor:	*Marc Lagunas Merino*
Nombre del consultor/a:	*Dr. Josep Vives Santa-Eulalia*
Nombre del PRA:	*Juan Alberto Rodríguez Velázquez*
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Resumen del Trabajo (máximo 250 palabras): *Con la finalidad, contexto de aplicación, metodología, resultados i conclusiones del trabajo.*

El trabajo que se presenta está enmarcado dentro de la teoría de procesos estocásticos aplicados a la valoración de opciones europeas. El objetivo del documento es presentar una nueva fórmula analítica para escribir la función de densidad del activo subyacente del derivado en términos de la ya conocida función de Black-Scholes más un término de ajuste que viene determinado por la correlación y la volatilidad que dependen del modelo de volatilidad estocástica escogido.

Para lograr llegar a esta nueva expresión analítica para un modelo de volatilidad estocástica generalizado se empieza desde un marco de valoración Black-Scholes con volatilidad constante y se evoluciona hacia un modelo de volatilidad estocástica introduciendo la teoría del cálculo de Malliavin que permite diferenciar funcionales en términos de un proceso de Wiener. Acto seguido, una vez presentado el nuevo marco de valoración con volatilidad estocástica y el propio proceso de volatilidad como un promedio de la volatilidad futura del subyacente, se demuestra el teorema que caracteriza un proceso anticipativo en términos de las derivadas de Malliavin del mismo proceso.

Finalmente una vez introducida esta teoría se presentan, por orden, la fórmula de Hull-White con correlación, que da una expresión del valor del derivado en términos de la función de Black-Scholes y una corrección por la correlación entre el proceso del subyacente y el proceso de volatilidad estocástica. A continuación, en el capítulo 3 se presenta el teorema de Breeden y

Litzenberger que relaciona el precio de una opción europea y la función de densidad del activo subyacente. Con esto, se acaba deduciendo una nueva expresión analítica para esa función de densidad en términos de la función de Black-Scholes y la corrección por correlación donde intervienen las derivadas de Malliavin y el proceso de volatilidad estocástica.

Abstract (in English, 250 words or less):

The following document belongs to the theory of stochastic processes applied to european option pricing. The goal of the whole project is deducing a new analytical formula that characterizes the underlying asset's probability density function in terms of the well known Black-Scholes formula plus a correction term that is determined by the correlation between the underlying's asset model and the generalized volatility model.

In an attempt to deduce the characterization, we start from a constant volatility Black-Scholes model and progressively modify it by replacing the constant volatility by a stochastic volatility model introducing the Malliavin Calculus theory, that will allow to differentiate smooth functionals with respect to the Wiener process. Once this theory is built and the volatility process is settled as an average of the underlying asset's future volatility, we will give a proof for the Theorem that gives a characterization of an anticipating process in terms of its Malliavin derivatives.

Now the following issues are introduced in order: the correlated version of Hull-White formula that characterizes a derivative's value in terms of the Black-Scholes formula plus a correction given by the correlation term. Finally in Chapter 3 the Breeden and Litzenberger Theorem is proved in order to relate the option's price and the underlying's asset probability density function. With all these tools we end up deducing the new analyitical expression that allow us to write the probability density function in terms of the Black-Scholes formula, the correlation term and the volatility's Malliavin derivative.

Contents

Chapter 1

Introduction

The following work is laid on the theory given by Fouque, Papanicolaou and Sircar in *Derivatives in Financial Markets with Stochastic Volatility*, see [6], and assuming the reader is familiar with stochastic processes, martingale theory, financial derivatives and Black-Scholes pricing theory. The later consists in a mathematical model of financial markets that gives investors a theoretical price of European-style options.

The model was first published in the Journal of Political Economy in a paper, see [4], written by Fischer Black and Myron Scholes in 1973, named *"The Pricing of Options and Corporate Liabilities".* Robert Merton, see [10], would be the first to publish a paper on the mathematical understanding of this pricing model. This mathematical theory granted them a Nobel Memorial Prize in Economic Sciences for their work in 1997, which was only given to Merton and Scholes since Fisher Black had already passed away at that time.

The model assumes that stock prices S_t satisfy the following stochastic differential equation (SDE).

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where μ and σ are constants and W_t is a standard Brownian motion. Now under a certain arbitrage-free market hypothesis, one could solve the SDE and write the price of the call option BS in an analytical formula as

$$BS(t, x; \sigma) := e^x \Phi(d_+) - Ke^{-r(T-t)} \Phi(d_-), \tag{1.2}$$

where $T - t$ is the option's time to maturity, x is the logarithm of the stock price S_t, K is the strike price, μ should be the risk-free rate r that ensures the discounted process is a martingale and Φ is the standard normal distribution of the quantities d_+ and d_- which are given by

$$d_{\pm} = \frac{x - ln(K) + (r \pm \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

Nowadays, it is widely known that if we take a collection of call options whose time to maturity is T and all are tied to the same underlying asset whose price is S_t, or equivalently $X_t = log(S_t)$, and different strike prices, we find out that despite having different market prices, they should all have the same constant volatility under the assumptions of Black-Scholes model, but in fact these volatilities that equate the respective market prices for each call option are not the same. This is widely known as the volatility smile effect, which breaks the Black-Scholes assumption of constant volatility.

A first approach to take into account the volatility smile when pricing under the Black-Scholes model is setting the volatility as a deterministic function of time, $\sigma = \sigma_t$. Under this new approach we can rewrite the underlying SDE equation (1.1) as

$$dS_t \quad = \quad \mu S_t dt + \sigma_t S_t dW_t, \tag{1.3}$$

Now for sake of simplicity in the next sections we will do an algebraic manipulation and write the previous expression in terms of the logarithmic price $X_t = log(S_t)$ under the risk-free measure, this is $\mu = r$, as

$$\frac{dS_t}{S_t} \quad = \quad rdt + \sigma_t dW_t,$$

Now applying the Itô Lemma[1], using $X_t = f(t, S_t) = log(S_t)$ as an auxiliary function, we have that

[1]See Appendix (A.1.1) for more detail on the Itô Lemma.

$$dX_t = \partial_t f dt + \partial_x f dS_t + \frac{1}{2}\left[\partial_{xx} f (dS_t)^2\right] \tag{1.4}$$

$$dX_t = rdt + \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt \tag{1.5}$$

and integrating both sides of the previous equation we have that we can write X_T as

$$X_T - X_t = \int_t^T (r - \frac{\sigma_s^2}{2})ds + \int_t^T \sigma_s^2 dW_s, \tag{1.6}$$

Now it is obvious that the following holds

$$X_T - X_t \sim \mathcal{N}((r - \frac{1}{2}\bar{\sigma}_t^2)(T-t), \bar{\sigma}_t^2(T-t)), \tag{1.7}$$

where

$$\bar{\sigma}_t := \sqrt{\frac{1}{T-t}\int_t^T \sigma_s^2 ds},$$

is commonly known as the future average volatility. Now we just have to replace the previous volatility constant in the Black-Scholes equation by the future average volatility $\bar{\sigma}_t$. Hence the derivative's value V_t, at time t, will be written as

$$V_t = BS(t, X_t, \bar{\sigma}_t), \tag{1.8}$$

since the process $\bar{\sigma}_t$ is anticipative because depends on future values of volatility. This is the main reason why we need to use the Malliavin Calculus tool in order to manage anticipative processes like Y_t defined by:

$$Y_t := (T-t)\bar{\sigma}_t^2 = \int_t^T \sigma_r^2 dr, \tag{1.9}$$

This kind of analytically tractable stochastic models are covered by A. Gulisashvili in [8], see also [7] for an introductory lecture on stochastic volatility models.

Chapter 2

Preliminaries on Malliavin Calculus

The following chapter is a brief introduction to the Malliavin Calculus. It covers basic definitions and propositions needed to manipulate stochastic anticipative proceses like volatility. We also build and prove the Hull-White formula that relates the price of an option in terms of the Black-Scholes function, plus a term that depends on the correlation and volatilities between the underlying asset and the volatility process. A more detailed lecture on topics related to Malliavin Calculus that are used in this chapter is [11].

2.1 Malliavin Derivatives and the Skorohod Integral

Let W be the canonical Wiener process, defined on the canonical space $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$[1], where $\Omega^W := C_0([0,T])$ is the space of continuous functions on $[0,T]$, null at the origin. Denote by $\mathbb{E}_W$ the expectation with respect to $\mathbb{P}_W$.

Let $H = L^2([0,T], \mathcal{B}([0,T]), l)$, be a Hilbert space where l denotes the Lebesgue measure on $[0,T]$ and denote by

[1] From now on, we will use indistinctly the following notation $(\Omega, \mathcal{F}, \mathbb{P})$ to refer to the canonical space adapted to the Wiener process.

$$W(h) := \int_0^T h(s)dW_s,$$

the Wiener integral of a deterministic function $h \in H$. Now consider the family of smooth functionals of type

$$F = f(W(h_1), \ldots, W(h_n)),$$

for any $n \geq 1$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$. Recall that a smooth functional is such that f and all its derivatives are bounded.

2.1.1 Definition. Given a smooth functional F, we define its Malliavin derivative, DF, as the element of $L^2(\Omega \times [0,T]^n)$ given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i(t), t \in [0,T]. \tag{2.1}$$

The operator D is a closable unbounded operator defined as

$$\begin{aligned} D. : L^2(\Omega) &\longrightarrow L^2([0,T]^n \times \Omega) \\ F = f(W(h_1), \ldots, W(h_n)) &\mapsto D.F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i(\cdot) \end{aligned}$$

Now we can define the iterated derivative operator as

$$D^n_{t_1,\ldots,t_n} = D_{t_1} \cdots D_{t_n} F.$$

This iterated derivative operator D^n is a closable unbounded operator from $L^2(\Omega) \to L^2([0,T] \times \Omega)$.

Now we can define the adjoint of the derivative operator, D, as δ, the Skorohod Integral. Let $Dom\delta$ be a set of elements $u \in L^2([0,T] \times \Omega)$, such that there exists a constant c satisfying

$$| \mathbb{E} \int_0^T D_t F u_t dt | \leq c||F||_{L^2(\Omega)}.$$

Now if $u \in Dom\delta$, then $\delta(u)$ is an element in $L^2(\Omega)$ characterized by

$$E\left(\delta(u)F\right) = E\int_0^T D_t F u_t dt. \tag{2.2}$$

We will make use of the notation $\delta(u) = \int_0^T u_t dW_t$ to refer to the Skorohod integral. Finally the following proposition will be used in the next section when treating anticipating processes. A full detailed proof can be found in [3].

2.1.2 Proposition. *Let $u \in Dom\delta$ and consider a random variable $F \in \mathbb{D}^{1,2}$, where $\mathbb{D}^{1,2}$ denotes the domain of the derivative operator and is a dense subset of $L^2(\Omega)$, such that $E(F^2\int_0^T u_t^2 dt) < \infty$. Then*

$$\int_0^T F u_t dW_t = F\int_0^T u_t dW_t - \int_0^T D_t F u_t dt, \tag{2.3}$$

where $Fu \in Dom\delta$ if and only if the right-hand side of (2.3) is square integrable.

2.1.3 Proposition. *Let $u \in L_a^2([0,T]\times\Omega)$, this is the set of square integrable adapted processes (with respect to the filtration generated by W). Then, for all $0 \le t < s \le T$*

$$D_s u_t = 0.$$

A detailed proof of this Proposition can be found in [11].

2.2 Itô formula for anticipating processes

This section is devoted to prove the following theorem that gives an extension of the Itô formula for anticipating stochastic processes, to do so we will follow the proof in [3].

2.2.1 Theorem. Consider a stochastic process of the form

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds,$$

where X_0 is a $\mathcal{F}_0-$mesurable random variable and both $u, v \in L_a^2([0,T] \times \Omega)$. Consider also a process $Y_t = \int_t^T \theta_s ds$, for some $\theta \in \mathbb{L}^{1,2}$, where $\mathbb{L}^{1,2}$ is the space of processes $u \in L^2([0,T] \times \Omega)$, such that $u_t \in \mathbb{D}^{1,2}$.

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated in (t, X_t, Y_t) are bounded by C. Then it follows that

$$\begin{aligned} F(t, X_t, Y_t) &= F(0, X_0, Y_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s, Y_s) ds \\ &+ \int_0^t \frac{\partial F}{\partial x}(s, X_s, Y_s) dX_s + \int_0^t \frac{\partial F}{\partial y}(s, X_s, Y_s) dY_s \\ &+ \int_0^t \frac{\partial^2 F}{\partial x \partial y}(s, X_s, Y_s)(D^- Y)_s u_s ds \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s, Y_s) u_s^2 ds, \end{aligned}$$

where $(D^- Y)_s := \int_s^T D_s Y_r dr$.

Proof: Fix $n \geq 1$ and let $t_i = it/n$. Under these conditions we apply the Taylor expansion up to the second order and obtain that

$$\begin{aligned}
F(t,X_t,Y_t) &= F(0,X_0,Y_0)+\sum_{i=0}^{n-1}\frac{\partial F}{\partial t}(t_i,X_{t_i},Y_{t_i})(t_{i+1}-t_i)\\
&+ \sum_{i=0}^{n-1}\frac{\partial F}{\partial x}(t_i,X_{t_i},Y_{t_i})(X_{t_{i+1}}-X_{t_i})\\
&+ \sum_{i=0}^{n-1}\frac{\partial F}{\partial y}(t_i,X_{t_i},Y_{t_i})(Y_{t_{i+1}}-Y_{t_i})\\
&+ \frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 F}{\partial t^2}(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})(t_{i+1}-t_i)^2\\
&+ \sum_{i=0}^{n-1}\frac{\partial^2 F}{\partial t\partial x}(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})(t_{i+1}-t_i)(X_{t_{i+1}}-X_{t_i})\\
&+ \sum_{i=0}^{n-1}\frac{\partial^2 F}{\partial t\partial y}(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})(t_{i+1}-t_i)(Y_{t_{i+1}}-Y_{t_i})\\
&+ \frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 F}{\partial x^2}(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})(X_{t_{i+1}}-X_{t_i})^2\\
&+ \sum_{i=0}^{n-1}\frac{\partial^2 F}{\partial x\partial y}(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})(X_{t_{i+1}}-X_{t_i})(Y_{t_{i+1}}-Y_{t_i})\\
&+ \frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 F}{\partial y^2}(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})(Y_{t_{i+1}}-Y_{t_i})^2\\
&= F(0,X_0,Y_0)\\
&+ T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8+T_9,
\end{aligned}$$

for some intermediate point $(\bar{t}_i,\bar{X}_{t_i},\bar{Y}_{t_i})$, between (t_i,X_{t_i},Y_{t_i}) and $(t_{i+1},X_{t_{i+1}},Y_{t_{i+1}})$. Now it is clear, by classical arguments, that

$$T_1\to\int_0^t\frac{\partial F}{\partial s}(s,X_s,Y_s)ds$$

and

$$T_3\to\int_0^t\frac{\partial F}{\partial y}(s,X_s,Y_s)\theta_s ds$$

in $L^1(\Omega)$.

For the addend T_2, we can write the following

$$\begin{aligned} T_2 &= \sum_{i=0}^{n-1} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i})(X_{t_{i+1}} - X_{t_i}) \\ &= \sum_{i=0}^{n-1} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \left[(X_0 + \int_0^{t_{i+1}} u_s dW_s + \int_0^{t_{i+1}} v_s ds) - (X_0 + \int_0^{t_i} u_s dW_s + \int_0^{t_i} v_s ds) \right] \\ &= \sum_{i=0}^{n-1} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \left(\int_{t_i}^{t_{i+1}} v_s ds \right) + \sum_{i=0}^{n-1} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \left(\int_{t_i}^{t_{i+1}} u_s dW_s \right). \end{aligned}$$

Again from classical arguments, is easy to see that

$$\sum_{i=0}^{n-1} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \left(\int_{t_i}^{t_{i+1}} v_s ds \right) \to_{L^1(\Omega)} \int_0^t \frac{\partial F}{\partial x}(s, X_s, Y_s) v_s ds.$$

Now using Proposition (2.1.2) it follows that

$$\begin{aligned} & \sum_{i=0}^{n-1} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \left(\int_{t_i}^{t_{i+1}} u_s dW_s \right) \\ = & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) u_s dW_s \\ + & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} D_s \left(\frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \right) u_s ds \end{aligned}$$

Using the chain rule for the derivative operator, see [11], and Proposition (2.1.3) we can deduce that

$$\int_{t_i}^{t_{i+1}} D_s \left(\frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) \right) u_s ds = \int_{t_i}^{t_{i+1}} \frac{\partial^2 F}{\partial x \partial y}(t_i, X_{t_i}, Y_{t_i})(D_s Y_{t_i}) u_s ds.$$

Using the same arguments we have that

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{\partial^2 F}{\partial x\partial y}(t_i,X_{t_i},Y_{t_i})(D_sY_{t_i})u_s ds \rightarrow_{L^1(\Omega)} \int_0^t \frac{\partial^2 F}{\partial x\partial y}(s,X_s,Y_s)(D^-Y)_s u_s ds,$$

and

$$T_7 \rightarrow_{L^1(\Omega)} \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s,Y_s)u_s^2 ds.$$

Since F and its derivatives are bounded by hypothesis, $\int_0^T Ev_s^2 ds < +\infty$ and $\int_0^T E(\theta_s^2)ds < \infty$ it follows that $T_4 + T_5 + T_6 + T_8 + T_9$ tends to zero in $L^1(\Omega)$.

Finally, we will prove that for all $t \in [0,T]$, the process

$$\frac{\partial F}{\partial x}(t_i,X_{t_i},Y_{t_i})u_s\mathbb{I}_{[0,T]}$$

belongs to the domain of δ and that

$$\begin{aligned}
&\int_0^t \frac{\partial F}{\partial x}(s,X_s,Y_s)u_s dW_s \\
= \; & F(t,X_t,Y_t) - F(0,X_0,Y_0) - \int_0^t \frac{\partial F}{\partial s}(s,X_s,Y_s)ds \\
- \; & \int_0^t \frac{\partial F}{\partial x}(s,X_s,Y_s)v_s ds - \int_0^t \frac{\partial F}{\partial y}(s,X_s,Y_s)dY_s \\
- \; & \int_0^t \frac{\partial^2 F}{\partial x\partial y}(s,X_s,Y_s)(D^-Y)_s u_s ds \\
- \; & \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s,Y_s)u_s^2 ds. \qquad (2.4)
\end{aligned}$$

Now applying Lemma 1 in [3] to the sequence of processes

$$\Phi_s^n = u_s \sum_{i=0}^{2^n-1}\frac{\partial F}{\partial x}(t_i,X_{t_i},Y_{t_i})\mathbb{I}_{[t_i,t_{i+1}]}(s),$$

we have that Φ^n converges in $L^2([0,T] \times \Omega)$ to $\Phi = u_s F'(t_i, X_{t_i}, Y_{t_i}) \mathbb{I}_{[0,t]}(s)$. as n tends to infinity. From the previous steps we have that

$$\delta(\Phi^n) = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \frac{\partial F}{\partial x}(t_i, X_{t_i}, Y_{t_i}) u_s dW_s$$

converges in $L^1(\Omega)$ to a random variable equal to the right-hand side of equation (2.4). Then in order to end the proof it suffices to prove that the right-hand side of equation (2.4) belongs to $L^2(\Omega)$. This follows easily from the hypotheses of the theorem. □

Chapter 3

The Hull and White formula.

It has already been proved that if σ is a stochastic process uncorrelated with price, *i.e.* $\rho = 0$, we have that the value of the derivative at time t is written as

$$V_t = \mathbb{E}_t(BS(t, X_t, \bar{\sigma}_t)),$$

where BS stands for the Black-Scholes formula in equation (1.8). This is the classical Hull-White formula that covers non correlated stochastic volatility models. Now since the future average volatility is an anticipative process, as proved in the previous section, we will make use of the Malliavin-Skorohod calculus tool to solve the case where volatility and the underlying are correlated. So following [12], see also [1], we will introduce the following theorem that allows us to write an option's value at time t as the Black-Scholes function plus a correction term by means of the correlation and stochastic volatility of the underlying asset.

3.0.2 Theorem. Assume that

– *(A1):* $\sigma^2 \in \mathbb{L}_W^{1,2}$,

– *(A2):* $\sigma \in \mathbb{L}_W^{1,2}$

Then we have that,

$$V_t = \mathbb{E}_t\left[BS(t, X_t, \bar{\sigma}_t)\right] + \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(T-t)}(\partial_x^3 - \partial_x^2)BS(s, X_s, \bar{\sigma}_s)\Lambda_s ds\right], \quad (3.1)$$

where

$$\Lambda_s := (\int_s^T D_s \sigma_r^2 dr)\sigma_s. \tag{3.2}$$

Proof: This proof is based on a *decomposition method* technique. Recall that

$$V_T = (e^{X_T} - K)^+ = BS(T, X_T, \bar{\sigma}_T),$$

hence

$$\begin{aligned} V_t &= e^{-r(T-t)} \mathbb{E}_t \left[BS(T, X_T, \bar{\sigma}_T) \right] \\ e^{-rt} V_t &= e^{-rT} \mathbb{E}_t \left[BS(T, X_T, \bar{\sigma}_T) \right] \\ e^{-rt} V_t &= \mathbb{E}_t \left[e^{-rT} BS(T, X_T, \bar{\sigma}_T) \right]. \end{aligned}$$

Now we will apply the previous anticipative Itô formula to the process

$$e^{-rs} BS(s, X_s, \bar{\sigma}_s),$$

between t and T, then take its conditional expectation $\mathbb{E}_t$ and multiply it by e^{rt}. Then we will obtain an expression for V_t. Now the anticipative Itô formula proved in section (2.1) from [3], can be adapted to our case. Define

$$Y_t := (T-t)\bar{\sigma}_t^2 = \int_t^T \sigma_r^2 dr. \tag{3.3}$$

Using *(A1),* we are under the conditions in Theorem (2.2.1) also found in [1], and so, for any smooth functional $F \in C_b^{1,2,2}([0,T] \times \mathbb{R} \times [0,\infty))$, we have

$$\begin{aligned} F(t, X_t, Y_t) &= F(0, X_0, Y_0) + \int_0^t \partial_s F(s, X_s, Y_s) ds + \delta_t^{W,B}(\partial_x F(\cdot, X_\cdot, Y_\cdot)\sigma_\cdot) \\ &+ \int_0^t \partial_x F(s, X_s, Y_s)(r - \frac{\sigma_s^2}{2}) ds - \int_0^t \partial_y F(s, X_s, Y_s)\sigma_s^2 ds \\ &+ \rho \int_0^t \partial_{xy} F(s, X_s, Y_s)\Lambda_s ds + \frac{1}{2}\int_0^t \partial_x^2 F(s, X_s, Y_s)\sigma_s^2 ds. \end{aligned}$$

Now we will apply this result to the function that concerns us, the Black-Scholes function. This is

$$F(s,x,y) := e^{rs}BS(s,x,\sqrt{\frac{y}{T-s}}). \tag{3.4}$$

The problem now is that this function doesn't satisfy the required conditions of the previous Itô formula since its derivatives are not bounded, so we will apply a small correction in the parameters of the previous function to overcome this problem. This is known as the *mollifier argument.* For $n \geq 1$ and $\delta > 0$, we will consider the following approximation,

$$F_{n,\delta}(s,x,y) := e^{-rs}BS(s,x,\sqrt{\frac{y+\delta}{T-s}})\phi(\frac{x}{n}), \tag{3.5}$$

where $\phi \in C_b^2(\mathbb{R})$, such that

$$\phi(z) = \begin{cases} 1 \text{ if } |z| \leq 1 \\ [0,1] \text{ if } |z| \in [1,2] \\ 0 \text{ if } |z| > 2 \end{cases} \tag{3.6}$$

Lets plot the previous function in order to recall that it is bounded as well as its derivatives.

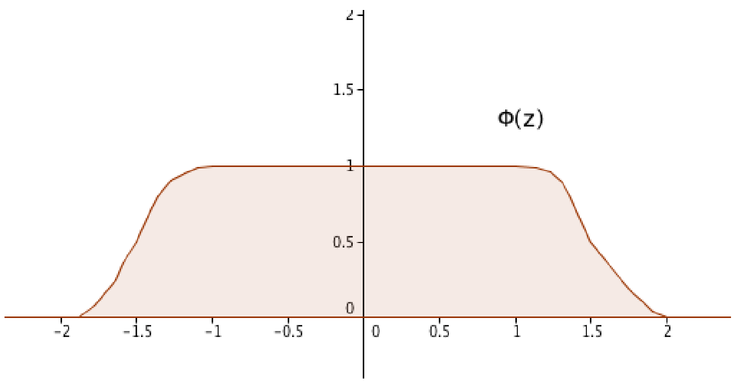

Figure 3.1: Adjustment function used to ensure all derivatives are bounded.

Then conditions to apply the anticipative Itô formula to $F_{n,\delta}(s, X_s, Y_s)$ are met. Taking the conditional expectation $\mathbb{E}_t$, using that Skorohod type integrals have zero expectation and multiplying by e^{rt}, we obtain

$$\begin{aligned}
& \mathbb{E}_t \left[e^{-r(T-t)} BS(T, X_T, \bar{\sigma}_T^\delta) \phi(\frac{X_T}{n}) \right] \\
= & \mathbb{E}_t \left[BS(t, X_t, \bar{\sigma}_t^\delta) \phi(\frac{X_t}{n}) \right] \\
+ & \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} A_n(s) ds \right] \\
+ & \frac{\rho}{2} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} (\partial_x^3 - \partial_x^2) BS(s, X_s, \bar{\sigma}_t^\delta) \phi_n(\frac{X_s}{n}) \Lambda_s ds \right] \\
+ & \frac{\rho}{2} \mathbb{E}_t \left[\int_t^T e^{-r(s-t)} (\partial_x^3 - \partial_x^2) BS(s, X_s, \bar{\sigma}_s^\delta) \frac{1}{n} \phi'(\frac{X_s}{n}) \Lambda_s ds \right],
\end{aligned}$$

where

$$\bar{\sigma}_s^\delta := \sqrt{\frac{Y_s + \delta}{T - s}},$$

and

$$\begin{aligned}
A_n(s) \quad := \quad & \frac{\sigma_s^2}{n} \partial_x BS(s, X_s, \bar{\sigma}_s^\delta) \phi'(\frac{X_s}{n}) \\
+ \quad & \frac{\sigma_s^2}{2n} BS(s, X_s, \bar{\sigma}_s^\delta) (\frac{1}{n} \phi''(\frac{X_s}{n}) - \phi'(\frac{X_s}{n})) \\
+ \quad & \frac{r}{n} BS(s, X_s, \bar{\sigma}_s^\delta) \phi'(\frac{X_s}{n}).
\end{aligned}$$

Finally, the result follows from the dominated convergence theorem, taking limits first on $n \uparrow +\infty$ and then on $\delta \downarrow 0$. The dominated convergence runs thanks to the properties of Black-Scholes function and $(A2)$. For the left hand side and the two first terms on the right hand side we use the fact tat function $BS(t, x, \sigma)$ is bounded by $e^x + K$ and its derivative $(\partial_x BS)(t, x, \sigma)$ is bounded by e^x. For the last two terms on the right hand side we use Lemma 2 in [2].

□

Chapter 4

Deriving the analytical form of the underlying probability density function.

The purpose of this section is to derive a new analytical form of writing the underlying probability density function in terms of the correlation between the underlying process and the stochastic volatility process and in terms of its Malliavin derivatives. To do so we need to introduce the so called Breeden and Litzenberger formula, see [5], that will allow us to relate the option's present value with the underlying asset's probability density function.

4.0.3 Theorem. Assume that the underlying asset, S_t, follows a Markov process with a probability density function $f_{S_T}(x)$ for S_T. Then it holds that

$$C_t(T,K) = e^{-r(T-t)} \int_K^\infty f_{S_T}(x)(x-K)dx \tag{4.1}$$

$$\frac{\partial C_t}{\partial K}(T,K) = -e^{-r(T-t)} \int_K^\infty f_{S_T}(x)dx \tag{4.2}$$

$$\frac{\partial C_t^2}{\partial K^2}(T,x) = e^{-r(T-t)} f_{S_T}(x). \tag{4.3}$$

Now we derivate the Hull-White formula for correlated stochastic volatility model with respect to K as it follows.

$$\frac{\partial^2 V_t}{\partial K^2} = \frac{\partial^2}{\partial K^2}\left(\mathbb{E}_t\left[BS(t,X_t,\bar{\sigma}_t)\right] + \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}(\partial_x^3 - \partial_x^2)BS(s,X_s,\bar{\sigma}_s)\Lambda_s ds\right]\right)$$

$$\begin{aligned}\frac{\partial^2 V_t}{\partial K^2} &= \mathbb{E}_t\left[\frac{\partial^2}{\partial K^2}(BS(t,X_t,\bar{\sigma}_t))\right] &(4.4)\\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(s-t)}(\partial_x^3 - \partial_x^2)\frac{\partial^2}{\partial K^2}(BS(s,X_s,\bar{\sigma}_s))\Lambda_s ds\right], &(4.5)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial}{\partial K}(BS(t,X_t,\bar{\sigma}_t)) &= \frac{\partial}{\partial K}(e^{X_t}\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-))\\ &= e^{X_t}\phi(d_+)\frac{\partial d_+}{\partial K} - e^{-r(T-t)}K\phi(d_-)\frac{\partial d_-}{\partial K} - e^{-r(T-t)}\Phi(d_-),\end{aligned}$$

since

$$\frac{\partial d_+}{\partial K} = \frac{-1}{K\bar{\sigma}_t\sqrt{T-t}} = \frac{\partial d_-}{\partial K},$$

we can rewrite the previous equation as it follows

$$\begin{aligned}\frac{\partial}{\partial K}(BS(t,X_t,\bar{\sigma}_t)) &= \frac{\partial d_+}{\partial K}(e^{X_t}\phi(d_+) - e^{-r(T-t)}K\phi(d_-)) - e^{-r(T-t)}\Phi(d_-)\\ &= -e^{-r(T-t)}\Phi(d_-),\end{aligned}$$

where $e^{X_t}\phi(d_+) - e^{-r(T-t)}K\phi(d_-) = 0$, (see A.1.2). Now for instance we will compute the second order derivative of $BS(t,X_t,\bar{\sigma}_t)$ with respect to K.

$$\begin{aligned}\frac{\partial^2}{\partial K^2}\left(BS(t,X_t,\bar{\sigma}_t)\right) &= \frac{\partial}{\partial K}\left(-e^{-r(T-t)}\Phi(d_-)\right)\\ &= -e^{-r(T-t)}\phi(d_-)\frac{\partial d_-}{\partial K}\\ &= \frac{e^{-r(T-t)}\phi(d_-)}{K\bar{\sigma}_t\sqrt{T-t}}\end{aligned}$$

$$\frac{\partial^2}{\partial K^2}(BS(t, X_t, \bar{\sigma}_t)) = \frac{e^{-r(T-t)}\phi(d_-)}{K\bar{\sigma}_t\sqrt{T-t}}. \tag{4.6}$$

Finally replacing the previous equation in equation (3.4) we have that.

$$\frac{\partial^2 V_t}{\partial K^2} = \mathbb{E}_t\left[\frac{e^{-r(T-t)}\phi(d_-)}{K\bar{\sigma}_t\sqrt{T-t}}\right] \tag{4.7}$$

$$+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(T-s)}(\partial_x^3 - \partial_x^2)\left(\frac{e^{-r(T-s)}\phi(d_-)}{K\bar{\sigma}_s\sqrt{T-s}}\right)\Lambda_s ds\right] \tag{4.8}$$

Now if the derivative we are pricing is a call option we have that $V_t = C_t$ and replacing the previous expression in equation (3.3) we can deduce by means of the Breeden-Litzenberger formula a new expression for the probability density function of $f_{S_T}(x)$

$$\begin{aligned}
f_{S_T}(x) &= e^{r(T-t)}\frac{\partial^2 V_t}{\partial K^2} \\
&= e^{r(T-t)}\mathbb{E}_t\left[\frac{e^{-r(T-t)}\phi(d_-)}{K\bar{\sigma}_t\sqrt{T-t}}\right] \\
&+ e^{r(T-t)}\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(T-s)}(\partial_x^3 - \partial_x^2)\left(\frac{e^{-r(T-s)}\phi(d_-)}{K\bar{\sigma}_s\sqrt{T-s}}\right)\Lambda_s ds\right] \\
&= \mathbb{E}_t\left[\frac{\phi(d_-)}{K\bar{\sigma}_t\sqrt{T-t}}\right] \\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(T-s)}(\partial_x^3 - \partial_x^2)\left(\frac{\phi(d_-)}{K\bar{\sigma}_s\sqrt{T-s}}\right)\Lambda_s ds\right]
\end{aligned}$$

Finally this can be rewritten as

$$\begin{aligned}
f_{S_T}(x) &= \mathbb{E}_t\left[\frac{\phi(d_-)}{K\bar{\sigma}_t\sqrt{T-t}}\right] \\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T \frac{e^{-r(T-s)}}{K\bar{\sigma}_s\sqrt{T-s}}(\partial_x^3 - \partial_x^2)\left(\phi(d_-)\right)\Lambda_s ds\right] \qquad (4.9)
\end{aligned}$$

4.0.4 Remark. Note that if σ is deterministic, the previous formula says that the law of S_T is lognormal, as expected. To prove so we will start by the uncorrelated case were volatility is constant and will try to derive the Black-Scholes underlying distribution. Now if $\rho = 0$ and $\bar{\sigma}_t = \sigma$ is constant we have that (4.9) is written as

$$f_{S_T}(x) = \frac{\phi(d_-)}{x\sigma\sqrt{\tau}},$$

where $\tau = T - t$. Lets start by proving that this follows a lognormal distribution by checking that

$$f_{S_T}(x) = \frac{\phi(d_-)}{x\sigma\sqrt{\tau}} \sim LN(\mu, \sigma^2)$$

In order to see that $f_{S_T}(x)$ follows a log-normal distribution as one would expect since we have come back to the basic model of Black-Scholes, we have to ensure that f_K fulfills the following

$$f \sim LN \quad \Leftrightarrow \quad f_{S_T}(x) = \frac{\phi(d_-)}{x\sigma\sqrt{\tau}} \sim LN \Leftrightarrow f_{S_T}(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}},$$

In order to simplify the calculus we will suppose, without loss of generality, that $T = 1$ and $t = 0$, hence $\tau = 1$. On the other hand recall that ϕ is the standard normal probability density function. Now all is left to do is checking the following equality

$$f_{S_T}(x) = \frac{\phi(d_-)}{x\sigma} \quad = \quad \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}} \tag{4.10}$$

$$\begin{aligned}
\Leftrightarrow \quad f_{S_T}(x) &= \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}} \\
\Leftrightarrow \quad f_{S_T}(x) &= \frac{1}{x\sigma\sqrt{2\pi}}exp\{\frac{-1}{2}\left[\frac{ln(S) - ln(K) + (r - \sigma^2/2)\tau}{2\sigma\sqrt{\tau}}\right]^2\} \\
\Leftrightarrow \quad f_{S_T}(x) &= \frac{1}{x\sigma\sqrt{2\pi}}exp\{\left[\frac{ln(K) - ln(S) - (r - \sigma^2/2)}{2\sigma^2}\right]^2\} \\
\Leftrightarrow \quad f_{S_T}(x) &= \frac{1}{x\sigma\sqrt{2\pi}}exp\{\frac{(ln(x) - \mu)^2)}{2\sigma^2},\}
\end{aligned}$$

where $\mu = (ln(S)+(r-\sigma^2/2)$. This ends the proof that the probability density function of the non correlated case follows a log-normal density function.

Now we will do the analogous task by proving that the underlying asset's probability density function is log-normal when considering the uncorrelated case and deterministic volatility instead of constant volatility as in the previous case.

We know from equation (1.6) that

$$X_T = X_t + \int_t^T (r - \frac{\sigma_s^2}{2})ds + \int_t^T \sigma_s dW_s,$$

Let us consider t=0 without loss of generality and the following stands.

$$\begin{aligned} X_T &= X_0 + \int_0^T (r - \frac{\sigma_s^2}{2})ds + \int_0^T \sigma_s dW_s \\ X_T - X_0 &= rT - \frac{1}{2}(\frac{1}{T}\int_0^T \sigma_s^2 ds)T + \frac{1}{T}(\int_0^T \sigma_s^2 ds)T \\ X_T - X_0 &\sim N(rT - \frac{1}{2}\bar{\sigma}_0^2 T, \bar{\sigma}_0^2 T). \end{aligned}$$

Recall that the future average volatility has already been defined as

$$\bar{\sigma}_t := \sqrt{\frac{1}{T-t}\int_t^T \sigma_s^2 ds}.$$

Now since X_T is the logarithm of the price process, S_T, and follows a Normal distribution, is obvious that S_T follows a Log-normal distribution.

Appendix A

A.1 Other definitions and concepts

A.1.1 Lemma. *The following Lemma is known as the Itô Lemma, see [9] for more detail. Let S_t be an Itô process given by an SDE of the following form*

$$dS_t = a(t, S_t)dt + b(t, S_t)dW_t,$$

and let X_t be another stochastic process written as a function of the previous process $f(t, S_t)$. Then X_t is also an Itô process given by

$$dX_t = \frac{\partial f}{\partial t}(t, S_t)dt + \frac{\partial f}{\partial x}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)(dS_t)^2.$$

A.1.2 Lemma.

$$S_t\phi(d_+) \quad = \quad e^{-r(T-t)}K\phi(d_-). \tag{A.1}$$

Proof:

$$\begin{aligned} & S_t\phi(d_+) = e^{-r(T-t)}K\phi(d_-) \\ \Leftrightarrow \quad & \frac{S_t}{K}e^{r(T-t)} = \frac{\phi(d_-)}{\phi(d_+)} \\ \Leftrightarrow \quad & log(\frac{S_t}{K}) + r(T-t) = \frac{d_+^2 - d_-^2}{2}, \end{aligned}$$

Now in addition we have that the right hand side of the last equation is

$$\begin{aligned}
\frac{d_+^2 - d_-^2}{2} &= \frac{1}{2}(d_+ + d_-)(d_+ - d_-) \\
&= \frac{1}{2}(2d_+ - \sigma\sqrt{T-t})\sigma\sqrt{T-t} \\
&= log(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t) - \frac{1}{2}\sigma^2(T-t) \\
&= log(\frac{S_t}{K}) + r(T-t).
\end{aligned}$$

□

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