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The Center-Focus and Ciclicity Problems: An Implementation of the Lyapunov Method and the Interpolation Technique

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FINAL MASTER PROJECT

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Abstract

The 16th Hilbert problem aims to determine the maximum number of isolated periodic solutions which a system of polynomial differential equations in the plane has. A first approach to this are the center-focus and ciclicity problems, which consist in identifying whether the origin of a system is a center or a focus and determining the maximum number of limit cycles, respectively. Here, we aim to study these problems for polynomial differential equations and this means to analyze the stability in a neighbourhood of a monodromic non-degenerate point. An essential mathematical object to deal with these issues are the Lyapunov quantities, which determine whether the origin is a center or a focus and its stability. In this project, we introduce two procedures to find these quantities: the Lyapunov method and an interpolation technique. Both algorithms are computationally implemented, and the differences between both methods are discussed. A parallelization example of the interpolation technique is also introduced, which shows the necessity of considering a parallel approach for the computation of the Lyapunov quantities in terms of efficiency and computational speed. Finally, the obtained code is used to analyze the center-focus and ciclicity problems for some particular polynomial differential equations: a cubic family, a quadratic family and a polyomial family with homogeneous nonlinearities example.

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1 Introduction and objectives

The purpose of this Final Master Project is to analyze the center-focus and ciclicity problems for several types of polynomial differential equations systems. This consists in studying the stability in a neighbourhood of the origin when it is a monodromic non-degenerate point –this is the name given to points which do not have any arrival direction, particularly those whose linear part has zero trace and positive determinant.

The center-focus problem, also known as the Poincaré center problem, consists in identifying whether the origin of a differential equations system in $\mathbb{R}^2$ whose origin is a monodromic non-degenerate point is a center or a focus. This means distinguishing whether the solutions near the origin are all periodic orbits or not. Solving this is a first step towards the ciclicity problem, which aims to determine the maximum number of limit cycles[1] which can appear when perturbing a system. All this is related to the 16th Hilbert problem, which according to [18] can be outlined as

"Proving that for any n≥2 there exists a finite number H(n) such that any polynomial differential equation whose degree is lower or equal than n has less than H(n) limit cycles."

This problem remains unsolved for most of the differential equations. Actually, the center-focus and ciclicity problems have been solved only for a few polynomial families. During this project some polynomial families for which the problem can be completed will be introduced.

Let us consider a system of differential equations in the plane such that its linear part has eigenvalues $\pm\alpha i$ with $\alpha \neq 0$. With an appropriate time change, we can assume $\alpha = 1$ and write the system in the form

$$\begin{cases} \dot{x} = -y + X(x, y, \lambda), \\ \dot{y} = x + Y(x, y, \lambda), \end{cases} \tag{1}$$

where $X(x, y, \lambda)$ and $Y(x, y, \lambda)$ are polynomials with degree at least 2 in x and y and parameters $\lambda = (\lambda_1, \ldots, \lambda_d)$ in the coefficients. As its linear part has eigenvalues $\pm i$, the Hartman Theorem cannot be applied for studying the system dynamics in a neighbourhood of the origin, so other techniques are required. A transformation into polar coordinates shows that if the origin is monodromic non-degenerate then the orbits near the origin spin around it, so the origin will be either a center or a focus.

To deal with this problem we first introduce the notion of Poincaré map ([23]). Let Σ

[1] Recall that a limit cycle is a periodic orbit which is isolated in the set of periodic orbits of a system

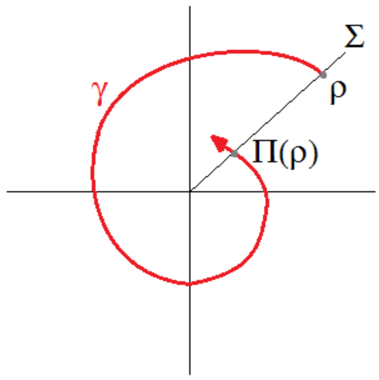


Figure 1: The Poincaré Map $\Pi(\rho)$.

be a section transversal to an orbit γ which is in a neighbourhood of the origin. The Poincaré map is a map $\Pi : \Sigma \to \Sigma$ such that, for an initial value $r(0) = \rho \in \gamma \cap \Sigma$ (where $r(\theta)$ is the radial coordinate of system (2) in polar coordinates), then $\Pi(\rho)$ is the first intersection point of orbit γ with Σ in positive time. This is schematized in Figure 1. One can prove that the Poincaré map can be analytically extended to $\rho = 0$, so we can consider its Taylor expansion

$$\Pi(\rho) - \rho = W_3\,\rho^3 + W_4\,\rho^4 + W_5\,\rho^5 + W_6\,\rho^6 + \cdots = \sum_{n=3}^{\infty} W_n\,\rho^n,$$

for certain values W_n which depend on parameters λ of (2). Observe that the center-focus problem is equivalent to determine whether all W_n are zero or not, since periodic orbits are fixed points of the Poincaré map. We will see that, if not all W_n vanish, the first non-zero W_n must have odd subindex. These W_n with odd $n \geq 3$ are known as Lyapunov quantities or Lyapunov constants, and they will be denoted as $W_n =: L_{(n-1)/2}$. In this case for which not all W_n vanish, the origin of the system is a focus and its stability is determined by the first non-zero Lyapunov quantity. As a consequence, the center-focus problem reduces to the problem of finding the Lyapunov quantities L_k.

This project will focus mainly on the efficient implementation of the computation of the Lyapunov quantities introduced above and its verification rather on the center-focus and ciclicity problems themselves. However, these problems are the context where the concept of Lyapunov quantities naturally arises, so it is necessary to carry out an accurate analysis of the problems. Furthermore, the implementation is used to solve the center-focus and ciclicity problems of some families of polynomial differential equations, both as a verification of the code and as an example of practical application.

The project will be divided into several sections. Firstly, the center-focus and ciclicity problems will be introduced with more detail, as well as some mathematical objects and results that will be useful. On the one hand, some ideas of ring theory will be reviewed since they are necessary for studying properties of the Lyapunov quantities. On the other hand, integrability and other tools to determine whether a system has a center at the origin will be studied, such as symmetries or Liénard systems properties. Even though this project is not about integrability, we will briefly introduce some basic ideas of Darboux integrability, a very useful theory in center characterization.

A second section will be dedicated to a deeper analysis of the Lyapunov quantities and their computation. Two procedures to find them will be explained: the Lyapunov method and the interpolation technique.

The next section will address the computational implementation of such methods. More precisely, both the Lyapunov method and the interpolation technique will be implemented in PARI/GP programming language. This will allow to deal with the center-focus and ciclicity problems in an easier and more efficient way. This implementation will be verified with some known examples in order to check that it works properly. In the last part of this section, an example to show the power of parallelizing the implemented code will be studied, and the advantages of parallelization and such an approach will be discussed.

Finally, the center-focus and ciclicity problems will be studied and solved for some concrete polynomial families. First, we will explicitly develop the method for a system of differential equations of cubic polynomials, which will be relatively simple. Later, quite a more complicated family of quadratic polynomials will be studied, and we will discuss why in this case the complexity in the resolution is increased. Finally, a last example about homogeneous nonlinearities in polynomial families will be analyzed.

It is worth remarking that this project is a first step towards the development of a PhD Thesis which I will start next year at UAB (Universitat Autònoma de Barcelona) under the supervision of Dr. Joan Torregrosa. He has also supervised this Final Master Project together with Dr. Villadelprat. I would like to thank Dr. Torregrosa for all the material he has provided me and the attention and help he has offered me. Some of the topics I deal with in this project are only a brief introduction of what I intend to do in the following research years. An example of this is the subsection about parallelization: here I will only introduce an example to show the necessity and power of parallelization in the resolution of our problem, while during my PhD I intend to develop a complete parallelization of all the implemented code.

With the contents explained here, the objectives which I expect to achieve are the following:

- **Assimilate and expand the mathematical knowledge learnt during the Master studies**. During this year, while studying the Master's Degree, I have learnt about many different topics on Computation Science and Mathematics. However, this is only a first step on the learning process, and with this project I expect to advance a new step and to link my Master studies with my future PhD. I intend to expand some notions and ideas which I already know, mainly in the field of Dynamical Systems but also using techniques of other subjects.

- **Develop research techniques**. Another capacity which a mathematician like me should have is the ability of searching information and understanding research papers. This is the reason why another objective I have during this project is to learn how to extract from articles and mathematical bibliography the information I need for my research, as well as to write it and communicate it with the appropriate scientific formalism.
- **Use computational mathematics as a support for the theoretical part**. Finally, with the codes which I will develop during the project I would like to show how computation can speed up mathematical procedures that otherwise would be inviable or very hard to execute.

The temporary planning I propose to achieve these objectives is shown in the Gantt diagram in Figure 2. It is worth noticing that the project will be developed during the summer months of 2017, this is approximately from mid-June to mid-September.

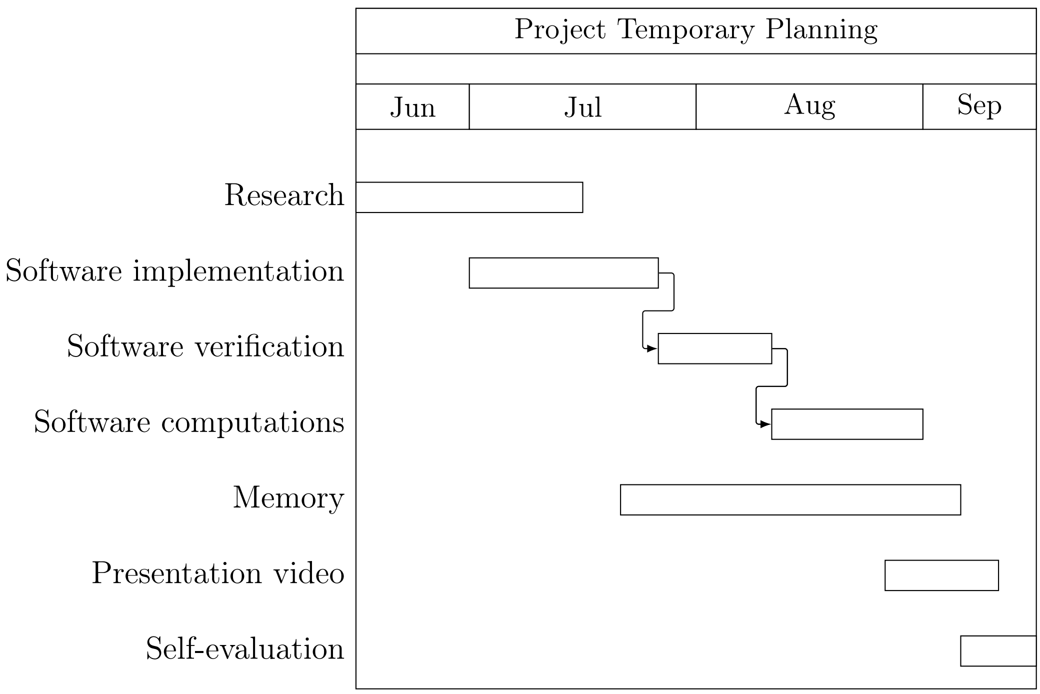


Figure 2: Temporary planning of the project.

2 Analysis of the problem

Let us consider a system of differential equations in the plane whose linear part has eigenvalues $\pm\alpha i$ with $\alpha \neq 0$. Performing a suitable time change we can assume $\alpha = 1$ and write the system in the form

$$\begin{cases} \dot{x} = -y + X(x,y), \\ \dot{y} = x + Y(x,y), \end{cases} \tag{2}$$

with $X(x,y)$ and $Y(x,y)$ such that

$$\begin{aligned} X(x,y) &= X_2(x,y) + X_3(x,y) + X_4(x,y) + \cdots , \\ Y(x,y) &= Y_2(x,y) + Y_3(x,y) + Y_4(x,y) + \cdots , \end{aligned}$$

where $X_k(x,y)$ and $Y_k(x,y)$ are homogeneous degree k polynomials with parameters $\lambda = (\lambda_1, \ldots, \lambda_d)$ in the coefficients, which means that the coefficients polynomially depend on λ. This will not be denoted as $X(x,y,\lambda)$ and $Y(x,y,\lambda)$ in order to simplify notation, but the dependence on parameters λ must be taken into account.

We are also interested in rewriting the system in complex coordinates $z = x + iy$. Let us derive z with respect to time t:

$$\dot{z} = \dot{x} + i\dot{y} = (-y + X(x,y)) + i\,(x + Y(x,y)) = i(x+iy) + (X(x,y) + iY(x,y)).$$

Let $w = x - iy$ be the conjugate of z. We define

$$Z(z,w) := X(x,y) + iY(x,y) = X\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) + iY\left(\frac{z+w}{2}, \frac{z-w}{2i}\right).$$

Then the system of differential equations (2) can be rewritten in complex variable as

$$\dot{z} = iz + Z(z,w), \tag{3}$$

which contains all the information of the problem. In this equation $Z(z,w)$ is such that

$$Z(z,w) = Z_2(z,w) + Z_3(z,w) + Z_4(z,w) \cdots , \tag{4}$$

with $Z_k(z,w)$ homogeneous degree k polynomials.

We aim to determine the stability of the origin for this type of systems. This is done by detecting whether the origin consists of a center or a focus, and by determining the maximum number of limit cycles which can appear when a perturbation occurs in a given degree family of polynomials.

2.1 Ideals and radicalness

In order to approach the center-focus and ciclicity problems we will need some objects and results from ring theory, which will be reviewed in this section.

Definition 2.1. *Let I be an ideal of a ring R. The radical of I is the set*

$$Rad\, I := \{r \in R \,|\, \exists\, n \in \mathbb{N},\, r^n \in I\}.$$

Definition 2.2. *An ideal I of a ring R is said to be radical if $I = Rad\ I$.*

Moreover, for our study we will need to use the Hilbert Basis Theorem, outlined as follows based on [8].

Theorem 2.3. ***(Hilbert Basis Theorem)*** *Let K be a field and we denote by $K[x_1, \ldots, x_n]$ the ring of polynomials with coefficients in K. Then every ideal I of $K[x_1, \ldots, x_n]$ is finitely generated, i.e. there exist $F_1, \ldots, F_s \in K[x_1, \ldots, x_n]$ such that $I = \langle F_1, \ldots, F_s \rangle$.*

Another important result which will be useful for us is the Hilbert Zeros Theorem (in German, *Hilberts Nullstellensatz*). This theorem requires some previous definitions.

Definition 2.4. *Given a set of points $U \in \mathbb{A}^n$, for a set $\mathbb{A}$, we define the ideal of U as*

$$\mathcal{I}(U) = \{F \in K[x_1, \ldots, x_n] \,|\, F(u) = 0 \text{ for all } u \in U\}.$$

It can be proved that $\mathcal{I}(U)$ is actually an ideal (see [8]).

Definition 2.5. *Given a field K and a set $\mathbb{A}$, for every ideal in $K[x_1, \ldots, x_n]$ the set of zeros or variety of I is*

$$V(I) = \{x \in \mathbb{A}^n(K) \,|\, P(x) = 0 \text{ for all } P \in I\}.$$

Now we can finally outline the theorem as it appears in [8].

Theorem 2.6. ***(Hilbert Zeros Theorem)*** *Let K be an algebraically closed field and let us denote by $K[x_1, \ldots, x_n]$ the ring of polynomials with coefficients in K. If I is an ideal of $K[x_1, \ldots, x_n]$, then*

$$\mathcal{I}(V(I)) = Rad\, I.$$

As a consequence, using this theorem and the definition of radical ideal we have the next result.

Corollary 2.7. *With the same notation as in Theorem 2.6, if I is a radical ideal then:*

$$\mathcal{I}(V(I)) = I.$$

2.2 Center-focus problem

Let us express system (2) in polar coordinates as follows:

$$\frac{dr}{dt} = r^2 P_2(\theta) + r^3 P_3(\theta) + \cdots , \tag{5}$$

$$\frac{d\theta}{dt} = 1 + r\, Q_2(\theta) + r^2\, Q_3(\theta) + \cdots , \tag{6}$$

where

$$P_i(\theta) = \cos\theta\, X_i(\cos\theta, \sin\theta) + \sin\theta\, Y_i(\cos\theta, \sin\theta),$$
$$Q_i(\theta) = \cos\theta\, Y_i(\cos\theta, \sin\theta) - \sin\theta\, X_i(\cos\theta, \sin\theta).$$

Now we divide equation (5) by (6),

$$\frac{dr}{d\theta} = \frac{r^2 P_2(\theta) + r^3 P_3(\theta) + \cdots}{1 + r\, Q_2(\theta) + r^2\, Q_3(\theta) + \cdots}. \tag{7}$$

We can observe that this function is analytic in a neighbourhood of the origin because the denominator does not vanish in $r = 0$. Let us then expand this function as a power series in r, which leads to an analytic differential equation,

$$\frac{dr}{d\theta} = R_2(\theta)\, r^2 + R_3(\theta)\, r^3 + \cdots , \tag{8}$$

for certain $R_i(\theta)$. This is an analytic differential equation with particular solution $r = 0$. Let $r(\theta, \rho)$ be the solution of equation (8) which $r(0, \rho) = \rho$. This solution is analytic in ρ, which is the initial value, so it can be expanded in the following way:

$$r(\theta, \rho) = \rho + u_2(\theta)\, \rho^2 + u_3(\theta)\, \rho^3 + \cdots . \tag{9}$$

As $r(0, \rho) = \rho$, it immediately follows that $u_k(0) = 0$ for every k. Let us study the stability near the origin, $r = 0$, by using

$$r(2\pi, \rho) = \rho + W_k\, \rho^k + O(k+1), \tag{10}$$

where $W_k := u_k(2\pi)$ is the first coefficient with does not vanish. Actually, the general definition is $W_j := u_j(2\pi)$ for $j \geq k$. The following result about these coefficients will be highly useful.

Lemma 2.8. *With the used notation, the first non-identically zero W_k has odd k. Furthermore, quantities W_j are polynomials in the coefficients of the original equation.*

We will not prove this statement, it has been extracted from [17]. Applying this Lemma, expression (10) can be rewritten as

$$r(2\pi, \rho) = \rho + W_{2n+1}\,\rho^{2n+1} + O(2n+2).$$

We will denote the W_{2n+1} with odd subscript as $W_{2n+1} =: L_n$; these values are well-known as Lyapunov quantities or Lyapunov constants. Actually, the Lyapunov quantity L_n is defined under the conditions on the parameters λ of the coefficients in (2) that $L_k = 0$ for $k < n$. Therefore, the problem reduces to study the Lyapunov quantities L_n. It is worth remarking, as the original equation consists of polynomials, all the implied functions involved the Lyapunov quantities will be analytic.

Observe that $r(2\pi, \rho)$ indicates the radius after a whole loop starting in the initial value ρ, so it turns out to be the Poincaré map defined in the introduction of the project:

$$\Pi(\rho) := r(2\pi, \rho) = \rho + W_3\,\rho^3 + W_4\,\rho^4 + W_5\,\rho^5 + W_6\,\rho^6 + \cdots = \rho + \sum_{n=3}^{\infty} W_n\,\rho^n.$$

Alternatively this can be written as

$$\Pi(\rho) - \rho = W_3\,\rho^3 + W_4\,\rho^4 + W_5\,\rho^5 + W_6\,\rho^6 + \cdots = \sum_{n=3}^{\infty} W_n\,\rho^n.$$

With this last expression of the Poincaré map, the next properties follow immediately.

Proposition 2.9. *The Poincaré map $\Pi(\rho)$ satisfies the following properties:*

(a) A certain initial condition ρ_0 defines a periodic orbit of system (2) if and only if $\Pi(\rho_0) - \rho_0 = 0$.

(b) Furthermore, this periodic orbit is a limit cycle if and only if ρ_0 is an isolated zero in the set of zeros of function $\Pi(\rho) - \rho$.

(c) The origin of the system (2) is a center if and only if $\Pi(\rho) - \rho \equiv 0$ in a neighbourhood of the origin.

According to property (c), solving the center-focus problem and characterizing a center is equivalent to determine under which conditions the Lyapunov quantities are $L_n = 0$ for all $n \geq 1$. There are two possible cases: either $L_n = 0$ for all n or there exists an n such that $L_n \neq 0$.

(i) If all L_n vanish, then $\Pi(\rho) - \rho \equiv 0$ so the origin is a center.

(ii) Otherwise, let L_l be the first non-zero Lyapunov quantity. In this case the origin is a focus, namely a weak focus of order l, and furthermore

- If $L_l < 0$ then it is an attracting focus (asimptotically stable).
- If $L_l > 0$ then it is a repelling focus.

These conclusions follow immediately from the Lyapunov stability Theorem, an elementary theorem in dynamical systems theory which can be found in [5].

Even though we will implement a program to compute the first N Lyapunov quantities, this is not enough to determine the centers. The reason is that finding that $L_n = 0$ for $n \leq N$ does not guarantee that all the Lyapunov quantities will vanish, and we would need to compute an infinite number of L_n.

Let us consider the ideal generated by all the Lyapunov quantities $\langle L_1, L_2, L_3, \ldots \rangle$, which according to Lemma 2.8 are polynomials. This is an ideal of the ring of polynomials $\mathbb{C}[\lambda]$, where $\lambda \in \mathbb{C}^d$ denotes the array of parameters in the coefficients of the differential system (3). Then, according to the Hilbert Basis Theorem, this ideal is finitely generated, so there must exist $m \in \mathbb{N}$ such that

$$\langle L_1, L_2, L_3, \ldots \rangle = \langle L_1, L_2, L_3, \ldots, L_m \rangle. \tag{11}$$

Knowing this m would significantly simplify the problem, because by computing the first m Lyapunov quantities (for example by using the program that will be implemented in this project) we would obtain center conditions. Nevertheless, as [4] states, there are no general methods find this m and this is the reason why the center-focus problem has been solved only for certain polynomial families.

2.3 Center characterization

As already stated, there are not general methods to find center conditions. Thus, in each particular case in which a point is a candidate to be a center, we will have to manage to see whether actually $L_n = 0$ for all $n \geq 1$. In the following subsections we will introduce some techiques which allow to do this: a theorem for characterizing centers in Liénard systems, a brief introduction to Darboux Integrability Theory, the use of symmetries to determine centers and a particular case of two monomial differential equations.

2.3.1 Liénard systems

In this subsection we will briefly introduce Liénard systems and a center characterization theorem for them, which will be useful because many other systems can be written as a Liénard system.

Definition 2.10. *A Liénard system is a ordinary system of differential equations in the plane which has the form*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - y\, f(x), \end{cases} \tag{12}$$

where f(x) and g(x) are real polynomials such that

$$g(0) = 0, \qquad g'(0) > 0.$$

Observe that a Liénard system with $g(x) = x + \tilde{g}(x)$, where $\tilde{g}(x)$ has at least degree 2, is a particular case of the system of equations (2) under a time change $t \to -t$.

Let $F(x)$ and $G(x)$ be the primitive functions of $f(x)$ and $g(x)$, respectively:

$$F(x) = \int_0^x f(s)\,ds \qquad G(x) = \int_0^x g(s)\,ds.$$

It is easy to see that, under the so-called Liénard transformation $(x, y) \to (x, y + F(x))$, system (12) can be rewritten as

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x). \end{cases} \tag{13}$$

Writing Liénard systems in this form will be helpful afterwards.

For Liénard systems, [6] outlines and proves the following necessary and sufficient center condition.

Theorem 2.11. *The origin of system (12) is a center if and only if $F(x) = \Phi(G(x))$ for any analytic function Φ such that $\Phi(0) = 0$.*

2.3.2 Darboux Integrability

Here we will see how the existence of first integrals is related to invariant curves through Darboux Integrability Theory. We denote $P := -y + X(x, y)$ i $Q := x + Y(x, y)$ in system (2). Let us first introduce the basic concepts of first integral and invariant algebraic curve.

Definition 2.12. ***(Integrability and first integral)*** *The polynomial system (2) is integrable on an open subset $U \subset \mathbb{R}^2$ if there exists a nonconstant analytic function $H : U \to \mathbb{R}$, called a first integral of the system on U, which is constant on all solution curves of system (2) contained in U.*

Definition 2.13. ***(Invariant algebraic curve)*** *Consider $f \in \mathbb{R}[x, y]$, where $\mathbb{R}[x, y]$ denotes the ring of polynomials in the variables x and y, f non-identically zero. The algebraic curve $f(x, y) = 0$ is an invariant algebraic curve of system (2) if*

$$(P, Q) \cdot \vec{\nabla} f = K f,$$

for some polynomial $K \in \mathbb{R}[x, y]$. The polynomial K is called the cofactor of the invariant algebraic curve $f = 0$. Finally, if f is an irreducible polynomial in $\mathbb{R}[x, y]$ we say that $f = 0$ is an irreducible invariant algebraic curve.

To study Darboux Integrability, we also need the notion of integrating factor.

Definition 2.14. ***(Integrating factor)*** *Let U be an open subset of $\mathbb{R}^2$ and let $R : U \to \mathbb{R}$ be an analytic function which is not identically zero on U. The function R is an integrating factor of system (2) on U if one of the following three equivalent conditions holds on U:*

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \operatorname{div}(RQ, RP) = 0, \quad P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -R\operatorname{div}(Q, P).$$

Definition 2.15. *The first integral H associated to the integrating factor R is given by*

$$H(x, y) = \int R(x, y)P(x, y)\,dy + \tilde{h}(x),$$

where $\tilde{h}$ is chosen such that $\frac{\partial H}{\partial x} = -RQ$; then

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \quad \dot{y} = RQ = -\frac{\partial H}{\partial x}.$$

There is another mathematical concept, the so-called exponential factor, which plays the same role as the invariant algebraic curves in obtaining a first integral of a polynomial system.

Definition 2.16. ***(Exponential factor)*** *Let h,$g \in \mathbb{R}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{R}[x, y]$ or that $h \equiv 1$. Then the function $\exp(g/h)$ is called an exponential factor of the system (2) if*

$$(P, Q) \cdot \vec{\bigtriangledown} \exp\left(\frac{g}{h}\right) = L \exp\left(\frac{g}{h}\right),$$

for some polynomial $L \in \mathbb{R}[x, y]$ of degree at most $\max\{\deg(P), \deg(Q)\}$-1. We say that the polynomial L is the cofactor of the exponential factor $\exp(g/h)$.

Now we can outline the following Darboux Integrability Theorem.

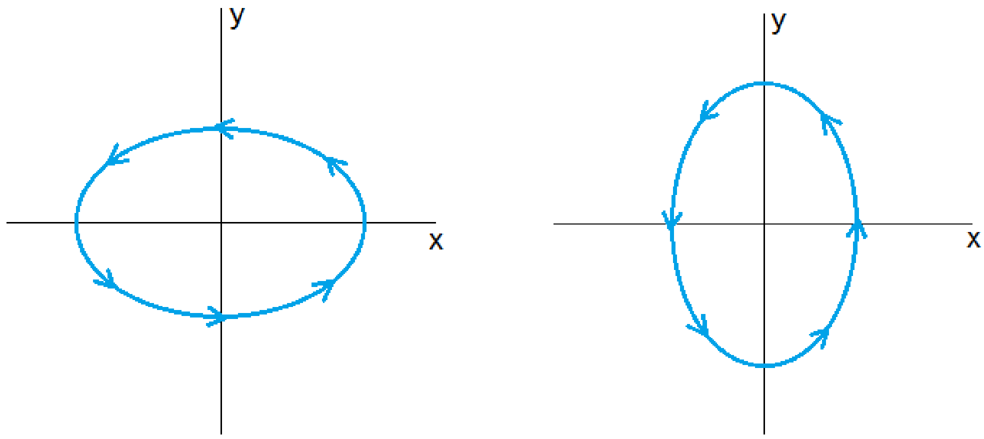
Theorem 2.17. ***(Darboux Integrability)*** *Suppose that system (2) admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$ and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$.*

(i) There exist λ_i,$\mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ if and only if the function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}, \tag{14}$$

is a first integral of system (2).

(ii) There exist λ_i,$\mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q)$ if and only if function (14) is an integrating factor of system (2).



(a) Symmetric with respect to the x axis. (b) Symmetric with respect to the y axis.

Figure 3: Periodic orbits which are symmetric with respect to the coordinate axes.

For a proof of this theorem see [9].

If a system has a focus on the origin then it cannot exist a first integral in a neighbourhood U of the origin. The reason is that if this first integral existed we would have that H is constant in all U, which contradicts the definition of first integral. As a consequence, in the center-focus problem analysis, if we manage to find a first integral –for example by means of Darboux Integrability Theorem– we can claim that the origin is a center.

2.3.3 Symmetries: straight lines reversibility

Another tool to determine whether a system has a center on the origin is identifying if there exists any kind of symmetry which allows to detect if orbits close on themselves or not. Even though there are many other, here we introduce only one example of symmetry: the so-called straight lines reversibility.

Let us consider a system which has the form (2) such that is invariant under a time and coordinates change $(x, y, t) \to (x, -y, -t)$. This means that the orbits near the origin are symmetric with respect to the x axis, so they follow a specific direction for $x > 0$ and the opposite direction but symmetrically for $x < 0$, as Figure 3a shows. This implies that all the orbits near the origin will close on themselves defining periodic orbits, which proves that the system has a center at the origin.

Otherwise, if the system remains invariant when applying a transformation $(x, y, t) \to (-x, y, -t)$ then the orbits near the origin are symmetric with respect to the y axis, as Figure 3b shows. Therefore, by an analogous reasoning as the previous case we deduce that this system also has a center at the origin.

2.3.4 Two monomial differential equations

In this subsection a center characterization technique is introduced for the family of differential equations

$$\dot{z} = iz + Az^k w^l + Bz^m w^n, \tag{15}$$

where $k+l \leq m+n$, $(k,l) \neq (m,n)$ and $A, B \in \mathbb{C}$. The integer values defined as

$$\alpha = k - l - 1, \quad \beta = m - n - 1,$$

will play a key role in our study. One of the reasons for this special role of both numbers is that when $\alpha = 0$ (resp. $\beta = 0$) the monomial $z^k w^l$ (resp. $z^m w^n$) appears as a resonant monomial.

The center characterization result we present here has been extracted from [10], and it is formulated as follows:

Theorem 2.18. *The origin of equation (15) is a center when one of the following (nonexclusive) conditions hold:*

(i) $k = n = 2$ *and* $l = m = 0$.

(ii) $l = n = 0$.

(iii) $A = -\bar{A}\, e^{i\alpha\varphi}$ *and* $B = -\bar{B}\, e^{i\beta\varphi}$ *for some* $\varphi \in \mathbb{R}$.

(iv) $k = m$ *and* $(l-n)\alpha \neq 0$.

Although these center conditions are only valid for the particular example of equations which have the form (15), they will be useful in some examples which we will analyze in later sections.

2.4 Ciclicity problem

On the other hand there is the ciclicity problem, which aims to determine the maximum number of limit cycles that can appear in a neighbourhood of the origin when slightly perturbing a system whose origin is a center or a focus. The problem can be outlined as follows: given a differential system which for a certain parameters λ_0 its origin is a monodromic non-degenerate point, which is the maximum number of limit cycles which will appear when perturbing the problem parameters? Applying Proposition 2.9, we can formally reformulate the problem as, given $\lambda_0 \in \mathbb{C}^d$ such that

$$\Pi(\rho, \lambda_0) - \rho = W_k \rho^k + \cdots, \tag{16}$$

with $W_k = L_{(k-1)/2} \neq 0$ (where k is odd) the first non-zero Lyapunov quantity, which is the maximum number of positive zeros that function $\Pi(\rho, \lambda) - \rho$ will have for $\lambda \sim \lambda_0$?

In order to solve the problem let us consider system (2) slightly perturbed with non-zero trace: we choose $a \sim 0$ but $a \neq 0$, this is, a arbitrarily close to 0, and rewrite the system as

$$\begin{cases} \dot{x} = ax - y + X(x,y), \\ \dot{y} = x + ay + Y(x,y). \end{cases}$$

Thus the linear part has trace $2a \sim 0$, $a \neq 0$. We proceed analogously to Section 2.2: rewrite the system in polar coordinates, and we obtain expressions which are similar to (5) and (6):

$$\frac{dr}{dt} = ar + r^2\,\widetilde{P_2}(\theta) + r^3\,\widetilde{P_3}(\theta) + \cdots\,,$$
$$\frac{d\theta}{dt} = 1 + r\,\widetilde{Q_2}(\theta) + r^2\,\widetilde{Q_3}(\theta) + \cdots\,.$$

Now divide both equations and expand in power series:

$$\frac{dr}{d\theta} = \frac{ar + r^2\,\widetilde{P_2}(\theta) + r^3\,\widetilde{P_3}(\theta) + \cdots.}{1 + r\,\widetilde{Q_2}(\theta) + r^2\,\widetilde{Q_3}(\theta) + \cdots} = ar + \widetilde{R_2}(\theta,a)\,r^2 + \widetilde{R_3}(\theta,a)\,r^3 + \cdots\,, \quad (17)$$

for certain $\widetilde{R_i}(\theta, a)$. Let $r(\theta, \rho, a)$ be the solution of (17) such that $r(0, \rho, a) = \rho$. This solution will be analyitc in ρ, which is the initial value, so it can be expanded as follows:

$$r(\theta,\rho,a) = e^{a\theta}\rho + \overline{u_2}(\theta,a)\,\rho^2 + \overline{u_3}(\theta,a)\,\rho^3 + \cdots\,.$$

- Let us set $L_k(a) := \overline{u_k}(2\pi, a)$, and we can rewrite the previous expression evaluating at 2π as

$$\Pi(\rho,a) - \rho = \left(e^{a2\pi} - 1\right)\rho + L_2(a)\,\rho^2 + L_3(a)\,\rho^3 + \cdots\,,$$

where $\Pi(\rho, a) := r(2\pi, \rho, a)$ is the Poincaré map. Notice that, as expected, if we set the trace to be zero, $a = 0$ and we obtain again expression (10) with $W_k = L_k(a = 0)$. If our differential equation has parameters $\lambda \in \mathbb{C}^d$, the trace a will depend on λ, $a = a(\lambda)$, so actually we can write $\Pi(\rho, \lambda)$ instead of $\Pi(\rho, a)$, where λ are the parameters of the original system. Assume that $a(\lambda_0) = 0$ for certain λ_0. Then, according to expression (16) we can write

$$\Pi(\rho,\lambda_0) - \rho = W_k\rho^k + \cdots\,.$$

If $W_k = L_{(k-1)/2} \neq 0$ (i.e., the origin is a focus), we aim to study what occurs when the system is perturbed, taking $\lambda \sim \lambda_0$. We define a distance function $f(\rho, \lambda) := \Pi(\rho, \lambda) - \rho$ and consider $\lambda \sim \lambda_0$:

$$f(\rho,\lambda) = \left(e^{a(\lambda)\,2\pi} - 1\right)\rho + L_2(\lambda)\,\rho^2 + L_3(\lambda)\,\rho^3 + \cdots\,, \quad (18)$$
$$f(\rho,\lambda_0) = W_k\rho^k + \cdots\,.$$

Notice that, if $\lambda = \lambda_0$ and so $a = 0$, all $L_k(a = 0) = W_k$ are polynomials in λ_0. Now we can apply on function $f(\rho, \lambda)$ the so-called Preparation Weierstrass, outlined as follows (it can be found in [21]).

Theorem 2.19. ***(Weierstrass Preparation Theorem)*** *Let f(x,λ) be an analytic function with x∈ ℂ and $\lambda \in \mathbb{C}^d$ near the origin. Let k be the lowest integer such that*

$$f(0,0) = 0,\ \frac{\partial f}{\partial x}(0,0) = 0,\ \ldots,\ \frac{\partial^{k-1} f}{\partial x^{k-1}}(0,0) = 0,\ \frac{\partial^k f}{\partial x^k}(0,0) \neq 0.$$

Then, near the origin, function f can be written in a unique way as a product of an analytic function c which is non-identically zero at the origin by an analytic function which consists of a degree k polynomial in x, i.e.

$$f(x,\lambda) = c(x,\lambda)\left(x^k + a_{k-1}(\lambda)x^{k-1} + \cdots + a_1(\lambda)x + a_0(\lambda)\right),$$

where functions c and a_i are analytic and c is non-identically zero at the origin.

When perturbing a focus, the first derivative of the distance function $f(\rho, \lambda)$ which does not vanish in $\rho = 0$, $\lambda = \lambda_0$ is that of order k, since $W_k = L_{(k-1)/2} \neq 0$. It is worth remarking that $L_{(k-1)/2} \neq 0$ because we are not perturbing a center. Therefore, applying the Weierstrass Preparation Theorem, we can rewrite $f(\rho, \lambda)$ as

$$f(\rho,\lambda) = c(\rho,\lambda)\left(\rho^k + a_{k-1}(\lambda)\rho^{k-1} + \cdots + a_1(\lambda)\rho\right),$$

for $(\rho, \lambda) \sim (0, \lambda_0)$ and where the involved functions are analytic. Notice that, according to (18), using the theorem notation we have $a_0(\lambda) = 0$.

As a consequence, the problem of finding the positive zeros of $f(\rho, \lambda)$ for $(\rho, \lambda) \sim (0, \lambda_0)$ reduces to finding the positive zeros of $\rho^k + a_{k-1}(\lambda)\rho^{k-1} + \cdots + a_1(\lambda)\rho$, since function $c(\rho, \lambda)$ does not vanish near λ_0. Our aim is then to solve equation

$$\rho^k + a_{k-1}(\lambda)\rho^{k-1} + \cdots + a_1(\lambda)\rho = 0. \tag{19}$$

It is trivial that one of the k solutions of (19) is $\rho = 0$. The following lemma, which is proved in [1], will be useful.

Lemma 2.20. *With the previous notation, the distance function satisfies $f(-\rho) = -f(\rho)$.*

Applying Lemma 2.20 we have that if $f(\rho_0) = 0$ then $f(-\rho_0) = 0$, so every positive solution is associated with a negative one and vice versa. As we also have $f(0) = 0$, we can conclude that the number of positive zeros is at most $(k-1)/2$ (recall that $k \geq 3$ is odd), and this will be the maximum number of limit cycles which can appear.

According to (11), there will be a certain m such that, if $L_n = 0$ for every $n \leq m$, we already have a center. In this case, $f(\rho, \lambda_0) \equiv 0$, so all the successive derivatives are also identically zero and the Preparation Weierstrass Theorem cannot be applied to complete the ciclicity study.

We will see in a later section an example of cubic polynomials, for which we will justify that if the ideal $\langle L_1, L_2, L_3, \ldots, L_m \rangle$ is radical then the problem can be solved in a relatively simple way. However, if the ideal is not radical the problem gets much more complicated, as happens for example with the quadratic polynomials that Bautin studied in [3] and which we will also analyze.

3 Lyapunov quantities computation

As introduced in the previous sections, the Lyapunov quantities are the main mathematical object to approach the center-focus and ciclicity problems. In this section we aim to present some methods which allow to compute these quantities, as well as their computational implementation.

The main technique we will see is the Lyapunov method, which uses Lyapunov functions[2] to find the quantities. This method will be introduced in subsection 3.1 and afterwards implemented. In the next subsection a new method for more complicated equations is presented. It is based on applying the previous Lyapunov method to some simple systems and using interpolation so as to obtain the Lyapunov quantities for a more complicated differential equation.

Apart from these methods we will see here, there are many other procedures to compute the Lyapunov quantities. For example, another interesting technique is the Andronov method. This method consists in writing the system in polar coordinates and, by means of derivation, obtain equivalent expressions whose coefficients can be equalized. This gives some integrals which can be solved in order to obtain the Lyapunov quantities. See [1] for more details on this method. It can be proved that Andronov method is equivalent to Lyapunov method, since stability and center conditions cannot depend on the used procedure.

3.1 The Lyapunov method

This method is based on the utilization of a Lyapunov function of system (2). The computation could be made using real values (see [22]), but the method is simplified if complex coordinates are considered. For this reason, in this project the method will be developed using complex variable. The objective is then to find a Lyapunov function F of system (3):

$$F = F_2 + F_3 + F_4 + \cdots , \tag{20}$$

with F_k an homogeneous degree k polynomial. Let us start with degree 2. We aim to study the sign of $\dot{F}$ and whether it vanishes or not. We compute

$$\dot{F} = F_z\, \dot{z} + F_w\, \dot{w} = F_z\, (iz + Z(z,w)) + F_w\, (-iw + \overline{Z(z,w)}) = \sum_{k\geq 1} L_k\, (zw)^{k+1}. \tag{21}$$

The last equality is a consequence of the following Theorem (extracted from [9]) applied on system (3):

[2]Given a differential system in $\mathbb{R}^n$ with and an open subset $U \subset \mathbb{R}^n$, we define a Lyapunov function (resp. a strict Lyapunov function) for an equilibrium point $x_0 \in \mathbb{R}^n$ as a scalar function $F : \mathbb{R}^n \to \mathbb{R}$ such that is continuous, has continuous derivatives, $F(x_0) = 0$ but $F(x) > 0$ locally near x_0 and $\dot{F}(X(t)) \leq 0$ (resp. $\dot{F}(x(t)) < 0$) locally near x_0.

Theorem 3.1. *Given the system of differential equations (2) (resp. (3)), if there exists a first integral F of the system then in suitable coordinates F is analytic on x^2+y^2 (resp. on zw). As a consequence, $\dot{F}$ is also analytic on x^2+y^2 (resp. on zw).*

Observe that, in expression (21), if all L_k vanish then $\dot{F}=0$, and therefore F is a first integral so the origin is a center. Otherwise, if any L_k is non-zero, according to Lyapunov Stability Theorem the origin will be a focus, either attracting or repelling according to the sign of the first non-zero L_k. Thus, these L_k are actually the Lyapunov quantities.

Now we will show how to find F recursively. It is worth remarking that the convergence of the series composed by all the found terms is an open problem yet. We impose equation (21) and perform formal operations as follows:

$$\begin{gathered}(F_{2z}+F_{3z}+F_{4z}+\cdots)\,(iz+Z_2+Z_3+Z_4+\cdots)+\\ +(F_{2w}+F_{3w}+F_{4w}+\cdots)\,(-iw+\overline{Z_2}+\overline{Z_3}+\overline{Z_4}+\cdots)=\\ =L_1\,(zw)^2+L_2\,(zw)^3+L_3\,(zw)^4+\cdots.\end{gathered}$$

Here we make terms having the same degree equal.

- Degree 2:
$$iz\,F_{2z}-iw\,F_{2w}=0\Longrightarrow z\,F_{2z}-w\,F_{2w}=0.$$
By deriving the corresponding polynomials and making their coefficients equal, it is easy to see that the solution to this equation is $F_2(z,w)=c\,zw$ for any constant $c\in\mathbb{C}$. We set $c=1/2$ and obtain
$$F_2(z,w)=\frac{zw}{2}.$$

- Degree 3:
$$iz\,F_{3z}-iw\,F_{3w}+Z_2\,F_{2z}+\overline{Z_2}\,F_{2w}=0.$$
The term $Z_2\,F_{2Z}+\overline{Z_2}\,F_{2w}$ is already known, so by writing F_3 as an homogeneous degree 3 polynomial in z and w and unknown coefficients, deriving and using the previous equation F_3 can be determined, if there exists a solution.

- Degree 4:
$$iz\,F_{4z}-iw\,F_{4w}+Z_3\,F_{2z}+\overline{Z_3}\,F_{2w}+Z_2\,F_{3z}+\overline{Z_2}\,F_{3w}=L_1\,(zw)^2.$$
And we operate analogously to the degree 3 case.

Using this reasoning and notation $\phi_{lk} := F_{lz}\, Z_k + F_{lw}\, \overline{Z_k}$ we can write the degree p equation as follows:

$$-iz\, F_{pz} + iw\, F_{pw} = \sum_{k=2}^{p-1} \phi_{p-k+1,k} - L_{\frac{p}{2}-1}\, (zw)^{\frac{p}{2}}, \tag{22}$$

with $L_{\frac{p}{2}-1} = 0$ if p is odd. As we will see, for even p the system matrix will have zero determinant. Therefore, so that the system has a solution we must force a suitable independent term so that the system is compatible. Now we will write this equation for degree p (22) in matrix form, and we will see how the whole problem can be reduced to solve a simple system of linear equations. Let us denote

$$F_p(z,w) := \sum_{j=0}^{p} h_{p-j,j}\, z^{p-j} w^j = h_{p,0}\, z^p + h_{p-1,1}\, z^{p-1} w + \cdots + h_{1,p-1}\, z w^{p-1} + h_{0,p}\, w^p.$$

Now we derive F_p with respect to z and w:

$$F_{pz} = \sum_{j=0}^{p} (p-j)\, h_{p-j,j}\, z^{p-j-1} w^j,$$

$$F_{pw} = \sum_{j=0}^{p} j\, h_{p-j,j}\, z^{p-j} w^{j-1}.$$

Thus,

$$\begin{aligned}
&- iz\, F_{pz} + iw\, F_{pw} = \\
&= -iz \left(\sum_{j=0}^{p} (p-j)\, h_{p-j,j}\, z^{p-j-1} w^j \right) + iw \left(\sum_{j=0}^{p} j\, h_{p-j,j}\, z^{p-j} w^{j-1} \right) = \\
&= -i \left(\sum_{j=0}^{p} (p-j)\, h_{p-j,j}\, z^{p-j} w^j \right) + i \left(\sum_{j=0}^{p} j\, h_{p-j,j}\, z^{p-j} w^j \right) = \\
&= i \sum_{j=0}^{p} \left(-(p-j)\, h_{p-j,j}\, z^{p-j} w^j + j\, h_{p-j,j}\, z^{p-j} w^j \right) = i \sum_{j=0}^{p} (2j-p)\, h_{p-j,j}\, z^{p-j} w^j.
\end{aligned}$$

Then, substituting in equation (22) we obtain

$$i \sum_{j=0}^{p} (2j-p)\, h_{p-j,j}\, z^{p-j} w^j = \sum_{k=2}^{p-1} \phi_{p-k+1,k} - L_{\frac{p}{2}-1}\, (zw)^{\frac{p}{2}},$$

and multiplying both members by $-i$ we finally obtain that equation (22) can be rewritten as

$$\sum_{j=0}^{p} (2j-p)\, h_{p-j,j}\, z^{p-j} w^j = -i \sum_{k=2}^{p-1} \phi_{p-k+1,k} + i\, L_{\frac{p}{2}-1}\, (zw)^{\frac{p}{2}}. \tag{23}$$

Our aim is to determine coefficients $h_{p-j,j}$ of term F_p which has degree p in the Lyapunov function F. Making coefficients equal in equation (23), we can outline a simple system of linear equations as follows.

- If p is odd, there is no $L_{\frac{p}{2}-1}$ and then the system can be written as

$$\begin{pmatrix} -p & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & -p+2 & 0 & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & & 0 & -1 & 0 & & & 0 \\ 0 & & & 0 & 1 & 0 & & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & & 0 & p-2 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & p \end{pmatrix} \begin{pmatrix} h_{p,0} \\ h_{p-1,1} \\ \vdots \\ h_{\frac{p+1}{2},\frac{p-1}{2}} \\ h_{\frac{p-1}{2},\frac{p+1}{2}} \\ \vdots \\ h_{1,p-1} \\ h_{0,p} \end{pmatrix} = \begin{pmatrix} \widetilde{\phi_0} \\ \widetilde{\phi_1} \\ \vdots \\ \widetilde{\phi_{\frac{p-1}{2}}} \\ \widetilde{\phi_{\frac{p+1}{2}}} \\ \vdots \\ \widetilde{\phi_{p-1}} \\ \widetilde{\phi_p} \end{pmatrix},$$

where the values $\widetilde{\phi_j}$ are the coefficients corresponding to $-i\sum_{k=2}^{p-1}\phi_{p-k+1,k}$ in equation (23) and they are known. Let us observe that in this case the system is compatible and determinate, and the values $h_{p-j,j}$ can be trivially computed as

$$h_{p-j,j} = \frac{\widetilde{\phi_j}}{2j-p}. \tag{24}$$

- If p is even, then the system can be written as

$$\begin{pmatrix} -p & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -p+2 & 0 & & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & & 0 & -2 & 0 & & & & 0 \\ 0 & & & 0 & 0 & 0 & & & 0 \\ 0 & & & & 0 & 2 & 0 & & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & & & 0 & p-2 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & p \end{pmatrix} \begin{pmatrix} h_{p,0} \\ h_{p-1,1} \\ \vdots \\ h_{\frac{p}{2}+1,\frac{p}{2}-1} \\ h_{\frac{p}{2},\frac{p}{2}} \\ h_{\frac{p}{2}-1,\frac{p}{2}+1} \\ \vdots \\ h_{1,p-1} \\ h_{0,p} \end{pmatrix} = \begin{pmatrix} \widetilde{\phi_0} \\ \widetilde{\phi_1} \\ \vdots \\ \widetilde{\phi_{\frac{p}{2}-1}} \\ \widetilde{\phi_{\frac{p}{2}}} + i\,L_{\frac{p}{2}-1} \\ \widetilde{\phi_{\frac{p}{2}+1}} \\ \vdots \\ \widetilde{\phi_{p-1}} \\ \widetilde{\phi_p} \end{pmatrix},$$

where components $\widetilde{\phi_j}$ are the coefficients corresponding to $-i\sum_{k=2}^{p-1}\phi_{p-k+1,k}$ in equation (23) and, as before, they are known. In the same way as in the odd case, coefficients $h_{p-j,j}$ for $j \neq \frac{p}{2}$ can be trivially determined using expression (24). Observe that for $j = \frac{p}{2}$ the equation

$$0\,h_{\frac{p}{2},\frac{p}{2}} = \widetilde{\phi_{\frac{p}{2}}} + i\,L_{\frac{p}{2}-1},$$

holds, so $h_{\frac{p}{2},\frac{p}{2}}$ remains as a free parameter, and for the sake of simplicity we set $h_{\frac{p}{2},\frac{p}{2}} = 0$. This equation also allows to find the Lyapunov quantity $L_{\frac{p}{2}-1}$:

$$0 = 0\, h_{\frac{p}{2},\frac{p}{2}} = \widetilde{\phi_{\frac{p}{2}}} + i\, L_{\frac{p}{2}-1} \Longrightarrow L_{\frac{p}{2}-1} = i\, \widetilde{\phi_{\frac{p}{2}}}.$$

Example 1

Let us consider the system of differential equations

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5, \\ \dot{y} = x. \end{cases}$$

We will illustrate how to use the explained Lyapunov method to find some coefficients of a Lyapunov function F of this system and its first Lyapunov quantity. Using the notation of equation (2), we have that $X(x,y) = a_2x^2+a_3x^3+a_4x^4+a_5x^5$ and $Y(x,y) = 0$. Writing in complex coordinates we obtain the system

$$\dot{z} = iz + a_2\left(\frac{z+w}{2}\right)^2 + a_3\left(\frac{z+w}{2}\right)^3 + a_4\left(\frac{z+w}{2}\right)^4 + a_5\left(\frac{z+w}{2}\right)^5,$$

and we have that $Z_i(z,w) = \overline{Z_i(z,w)} = a_i\left(\frac{z+w}{2}\right)^i$ for $i = 2,3,4,5$.

- Degree 2. We have seen that the degree 2 term of the Lyapunov function F (expression (20)) is

$$F_2 = \frac{zw}{2}.$$

- Degree 3. We set out the degree 3 term of the Lyapunov function F as

$$F_3 = h_{30}z^3 + h_{21}z^2w + h_{12}zw^2 + h_{03}w^3,$$

where $h_{kj} = a_{kj} + ib_{kj}$. We can find ϕ_{22} by doing

$$\phi_{22} = F_{2z}Z_2 + F_{2w}\overline{Z_2} = \frac{w}{2}a_2\left(\frac{z+w}{2}\right)^2 + \frac{z}{2}a_2\left(\frac{z+w}{2}\right)^2 =$$
$$= a_2\frac{z+w}{2}\left(\frac{z+w}{2}\right)^2 = a_2\left(\frac{z+w}{2}\right)^3.$$

With this we can write the matrix system introduced in the method explanation, and we can then compute the coefficients h_{kj} by means of expression (24):

$$\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} h_{30} \\ h_{21} \\ h_{12} \\ h_{03} \end{pmatrix} = \begin{pmatrix} -i\frac{1}{8}a_2 \\ -i\frac{3}{8}a_2 \\ -i\frac{3}{8}a_2 \\ -i\frac{1}{8}a_2 \end{pmatrix},$$

$$h_{30} = i\frac{1}{24}a_2, \qquad h_{21} = i\frac{3}{8}a_2, \qquad h_{12} = -i\frac{3}{8}a_2, \qquad h_{03} = -i\frac{1}{24}a_2. \quad (25)$$

Then,

$$F_3 = i\frac{a_2}{24}\left(z^3 + 9z^2w - 9zw^2 - w^3\right).$$

- Degree 4. The degree 4 term of F is

$$F_4 = h_{40}z^4 + h_{31}z^3w + h_{22}z^2w^2 + h_{13}zw^3 + h_{04}w^4.$$

Let us find ϕ_{23} and ϕ_{32}:

$$\begin{aligned}
\phi_{23} =& F_{2z}Z_3 + F_{2w}\overline{Z_3} = \frac{w}{2}a_3\left(\frac{z+w}{2}\right)^3 + \frac{z}{2}a_3\left(\frac{z+w}{2}\right)^3 = a_3\left(\frac{z+w}{2}\right)^4, \\
\phi_{32} =& F_{3z}Z_2 + F_{3w}\overline{Z_2} = \\
=& i\frac{a_2}{24}(3z^2 + 18zw - 9w^2)a_2\left(\frac{z+w}{2}\right)^2 + \\
&+ i\frac{a_2}{24}(9z^2 - 18zw - 3w^2)a_2\left(\frac{z+w}{2}\right)^2 = \\
=& i\frac{{a_2}^2}{24}(12z^2 - 12w^2)\left(\frac{z+w}{2}\right)^2 = i{a_2}^2(z-w)\left(\frac{z+w}{2}\right)^3.
\end{aligned}$$

Therefore the degree 4 system can be written as follows and the coefficients can be found:

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} h_{40} \\ h_{31} \\ h_{22} \\ h_{13} \\ h_{04} \end{pmatrix} = \begin{pmatrix} \frac{1}{8}{a_2}^2 - \frac{1}{16}ia_3 \\ \frac{1}{4}{a_2}^2 - \frac{1}{4}ia_3 \\ iL_1 - \frac{3}{8}ia_3 \\ -\frac{1}{4}{a_2}^2 - \frac{1}{4}ia_3 \\ -\frac{1}{8}{a_2}^2 - \frac{1}{16}ia_3 \end{pmatrix},$$

$$\begin{gathered}
h_{40} = -\frac{1}{32}{a_2}^2 + \frac{1}{64}ia_3, \qquad h_{31} = -\frac{1}{8}{a_2}^2 + \frac{1}{8}ia_3, \\
h_{22} = 0 \text{ (we impose)}, \\
h_{13} = -\frac{1}{8}{a_2}^2 - \frac{1}{8}ia_3, \qquad h_{04} = -\frac{1}{32}{a_2}^2 - \frac{1}{64}ia_3, \\
iL_1 - \frac{3}{8}ia_3 = 0 \Longrightarrow L_1 = \frac{3}{8}a_3.
\end{gathered} \quad (26)$$

Then the coefficients of the degree 4 terms of the Lyapunov quantity have been obtained, as well as the first Lyapunov quantity.

3.2 Lyapunov quantities via interpolation

Let us consider the system of differential equations in the complex plane (3),

$$\dot{z} = iz + Z(z, w),$$

where $Z(z, w)$ is such that

$$Z(z, w) = Z_2(z, w) + Z_3(z, w) + Z_4(z, w) \cdots ,$$

and every $Z_k(z, w)$ is an homogeneous degree k polynomial with the form $Z_k(z, w) = \sum_{j=0}^{k} r_{k-j,j} z^{k-j} w^j$, where $r_{k-j,j} \in \mathbb{C}$ is the corresponding coefficient. Recall that w denotes the conjugate of z, this is $w = \overline{z}$.

Before stating our results and introducing our method we need some definitions.

Definition 3.2. *M is a monomial of (3) when $M = \prod_{k,l} r_{k,l}^{m_{k,l}} \overline{r_{k,l}}^{n_{k,l}}$, with $m_{k,l}, n_{k,l} \in \mathbb{N}$, where the product is finite and $r_{k,l}$ is any coefficient of $Z_{k+l}(z, w)$.*

Definition 3.3. *A monomial M of (3) is a monomial of the n-th Lyapunov quantity L_n if the expression of the quantity L_n has a term with either* Re*(M) or* Im*(M).*

Definition 3.4. *Let M be a monomial as defined above. We define the degree, deg(M), the quasi degree, qd(M), and the weight of M, w(M), respectively, as*

$$deg(M) = \sum_{k,l} (m_{k,l} + n_{k,l}), \tag{27}$$

$$qd(M) = \sum_{k,l} (k + l - 1)(m_{k,l} + n_{k,l}), \tag{28}$$

$$w(M) = \sum_{k,l} (1 - k + l)(m_{k,l} - n_{k,l}). \tag{29}$$

Definition 3.5. *We say that a monomial of weight zero, M, is basic if M'|M and w(M')=0 imply that $M' = \pm M$. In other words, the basic monomials are the prime factors of the monomials of weight zero.*

With the above notation, the following result is well known; see [7], [15], [16], [19] and [24].

Theorem 3.6. *Let M be a monomial of the Lyapunov quantity L_n. Then qd(M)=2n and w(M)=0.*

This theorem gives some information about the monomials that appear in the Lyapunov quantities. [11] proves a result which improves Theorem 3.6 by describing how these monomials are distributed according to their degrees. This improvement is due to the fact that the new theorem restricts even more the monomials which can appear in the Lyapunov quantity. The result is as follows.

Theorem 3.7. *Let $M_1, \ldots, M_k$ be monomials of L_n with even degree, and $M_{k+1}, \ldots,$ M_{k+l} monomials of L_n with odd degree. Then*

$$L_n = \sum_{i=1}^{k} \alpha_i \mathrm{Im}(\mathrm{M_i}) + \sum_{\mathrm{i=k+1}}^{\mathrm{k+l}} \beta_\mathrm{i} \mathrm{Re}(\mathrm{M_i}) \tag{30}$$

for some $\alpha_i, \beta_i \in \mathbb{R}$.

Observe that, for any differential equation, the Lyapunov quantities L_n are real numbers. Hence, if M is a monomial of (3), $L_n = \alpha M + \overline{\alpha} \overline{M} + N$, where N denotes the sum of the other monomials appearing in the expression, and so $L_n = 2\mathrm{Re}(\alpha)\mathrm{Re}(M) - 2\mathrm{Im}(\alpha)\mathrm{Im}(M) + N$. Therefore, Theorem 3.7 reduces by half the estimation of the length of the Lyapunov quantities obtained using only Theorem 3.6.

We present now a method to compute the general formula of the Lyapunov quantities via interpolation. Let us suppose that we want to find the expression of the n-th Lyapunov quantity L_n for a differential equation of type (3). We proceed as follows:

1. By using Theorems 3.6 and 3.7 we list all the monomials involved in L_n. That is, we write L_n as a linear function of products of basic monomials and their unknown coefficients.

2. Once the monomials are listed, we look for all the undetermined coefficients by computing the Lyapunov quantity for some particular systems. This can be done by applying the Lyapunov mehtod introduced in the previous section. Then, by interpolation, we obtain the general expression of the constant L_n.

This method will be computationally implemented and illustrated with some examples in the following sections.

Finally, we briefly justify the use of interpolation. One can wonder why to use interpolation if the Lyapunov method from the previous section already computes the Lyapunov quantities. The answer is that, for some cumbersome differential equations, or polynomial equations whose coefficients contain many parameters, using the Lyapunov method can be slow and inefficient, and even some problems could not be solved. In this case, the interpolation technique described here reduces the problem to apply the Lyapunov method to many simple differential equations adequately chosen, and then find the Lyapunov quantity for the original equation. Therefore, a complicated and slow problem can be split into several simple and fast problems which can even be parallelized; then, when these simple problems have been solved, the solution of the initial problem can be found by means of interpolation.

4 Computational calculation of the Lyapunov quantities

The purpose of this section is to develop the computational implementation of the two procedures which compute the Lyapunov quantities previously introduced: the Lyapunov method and the interpolation technique. We will also see a first step to the parallelization of this implementation, and discuss its advantages.

4.1 Programming language choice

First, a programming language must be selected. From the technical point of view, a language which is relatively powerful and fast when operating would be good. After considering many options, the final choice is to use the software PARI/GP, or simply PARI. PARI is a specialized computer algebra system which, according to its creators, is designed to users whose primary need is speed, since its main advantage is execution velocity. This software is suitable when working with rational numbers with a lot of digits, and works properly when not much polynomial algebra is needed. However, although quite an amount of symbolic manipulation is possible, PARI does badly compared to systems like Maple. This is the reason why some of the outputs of our PARI codes will be afterwards treated with Maple.

With the help of the PARI/GP guide [2] the code to compute the Lyapunov quantities will be implemented in PARI. This code will be contained in a PARI/GP file named `lyapunov_quantities.gp`, attached to this project, and it is also shown in Appendix A. In the following subsections, the elaboration of this code will be explained with more detail.

4.2 Implementation of the Lyapunov method

The part of the code in file `lyapunov_quantities.gp` and in Appendix A which implements in PARI the Lyapunov method consists of the following four functions:

- Function ***complex(X, Y)***. Given a differential equations system as (2), this function returns a polynomial which represents the same system in complex coordinates. The function arguments X and Y correspond respectively to polynomials *X(x,y)* and *Y(x,y)* of system (2). The function returns the polynomial Z, where $z = x + iy$ and $w = \bar{z}$, corresponding to $\dot{z}$ as follows:

$$\dot{z} = \dot{x} + i\dot{y} = [-y + X(x,y)] + i\,[x + Y(x,y)] = -iz + [X(x,y) + iY(x,y)]\,,$$
$$Z := -iz + \left[X\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) + iY\left(\frac{z+w}{2}, \frac{z-w}{2i}\right)\right].$$

  Observe that if the system we need to study is already written in complex coordinates this function will not be required.

- Function ***conjugate(Z)***. Its utility is to conjugate any expression Z which is passed as an argument. In particular, it swaps from z to w and vice versa, since $w = \bar{z}$. Furthermore, the PARI instruction *conj* is applied on the result so that I is substituted by $-I$ and vice versa, which completes the conjugation. This function is always necessary because the main routine *lyapunov_method* calls it.

- Function ***complex_product(a,b)***. This function implements the product between two numbers or polynomials a and b which contain complex values. It is mandatory to define this function because the main routine *lyapunov_method* uses it.

- Function ***lyapunov_method(Z,M)***. This is the main function in the code, since it performs all the operations described in Section 3.1 to compute the Lyapunov quantities. The first argument Z is the polynomial in complex coordinates which describe the differential equation to analyze. If the given equation is a system of two real differential equations, this argument Z will result of evaluating this system on function *complex*. Argument M is the number of Lyapunov quantities which we want the program to find.

  The code uses an auxiliary matrix h which stores the coefficients of the Lyapunov function (20), so coefficient (i, j) of degree $i+j$ in the Lyapunov function corresponds to component $h[i+1, j+1]$ of the matrix[3]. Furthermore we define a vector L which stores the Lyapunov quantities, and this is the return value of the function. Two local variables are declared: p, which refers to the current degree in the calculations sequence, and ϕ, which contains the expression that is denoted as $\phi_{p-k+1,k}$ in (23). The function performs a loop which increases the degree p applying the described Lyapunov method algorithm, while it fills the Lyapunov quantities vector L, until reaching the required number of quantities M.

Finally, it is worth making the following observation. The parameters on the coefficients in the original differential equation can be complex values. However, due to the way that function *conjugate* works, if one needs that these coefficients are actually treated as complex values it is necessary to split real and imaginary part. Otherwise, function *conjugate* will not conjugate these parameters because it does not consider them to be complex. In particular, if we have a parameter $\lambda_j \in \mathbb{C}$ in any coefficient and we want it to be treated as a complex number, we will introduce

[3]Let us justify that matrix h has size $(2M + 3) \times (2M + 3)$. Observe that, for even degree p, we obtain the Lyapunov quantity $L_{\frac{p}{2}-1}$, as seen previously. Thus, if we want to compute the M-th quantity we will have $M = \frac{p_{max}}{2} - 1$, this is $p_{max} = 2M + 2$. Notice that the matrix size must be $(p_{max}+1)\times(p_{max}+1)$, since all the degrees p_{max} must be stored plus one corresponding to the first component (degree 0). Therefore, as $p_{max} = 2M+2$, the matrix size must be $(2M+3)\times(2M+3)$.

the polinomial which defines the equation as $a_j + ib_j$ instead of λ_j for certain a_j and b_j, so that $a_j, b_j \in \mathbb{R}$ and there is no inconvenience when conjugating.

4.2.1 Verification of the code with some examples

Example 1

Recall the example we saw in Section 3.1:

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5, \\ \dot{y} = x. \end{cases}$$

We computed the first Lyapunov quantity and the coefficients of the degree 3 and 4 terms of the Lyapunov function in expressions (25) and (26), obtaining

$$h_{30} = i\frac{1}{24}a_2, \qquad h_{21} = i\frac{3}{8}a_2, \qquad h_{12} = -i\frac{3}{8}a_2,$$
$$h_{03} = -i\frac{1}{24}a_2, \qquad h_{40} = -\frac{1}{32}{a_2}^2 + \frac{1}{64}ia_3, \qquad h_{31} = -\frac{1}{8}{a_2}^2 + \frac{1}{8}ia_3,$$
$$h_{22} = 0, \qquad h_{13} = -\frac{1}{8}{a_2}^2 - \frac{1}{8}ia_3, \qquad h_{04} = -\frac{1}{32}{a_2}^2 - \frac{1}{64}ia_3,$$
$$L_1 = \frac{3}{8}a_3.$$

Now we want to perform the same computation using the elaborated code, and check that tha same result is obtained to verify that the program works properly. The execution in the PARI/GP calculator is as follows:

```
? X=a2*x^2+a3*x^3+a4*x^4+a5*x^5;
? Y=0;
? Z=complex(X,Y);
? lyapunov_method(Z,1);
? L
%21 = [3/8*a3]
? h
%22 =
[                      0                          0          0           -1/24*I*a2 -1/32*a2^2 - 1/64*I*a3]

[                      0              1/2 -3/8*I*a2 -1/8*a2^2 - 1/8*I*a3                               0]

[                      0                  3/8*I*a2          0                    0                    0]

[             1/24*I*a2 -1/8*a2^2 + 1/8*I*a3          0                    0                               0]

[-1/32*a2^2 + 1/64*I*a3                           0          0                    0                    0]
```

As coefficient h_{ij} in the Lyapunov function corresponds to the component $(i+1, j+1)$ from matrix h, we see that the results obtained by executing the code for these coefficients and for L_1 are the same that those we obtained by manually applying the method.

Example 2

Let us study a second example, extracted from [14]. The system is

$$\dot{z} = iz - \frac{n}{n-2} z^n + zw^{n-1} + i\tau_n w^n. \tag{31}$$

Paper [14] states that for $n = 6, 8, 10, 12, 14, 16, 18$ there exists values τ_n such that equation (31) has a weak focus of order $n^2 + n - 2$ at the origin. Let us check this for $n = 6$:

```
? n=6;
? lyapunov_method(I*z-(n/(n-2))*z^n+z*w^(n-1)+I*tn*w^n,n^2+n-2);
? L
%25 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
, 0, 0, 0, 0, 0, 0, 0, 0, 0, -41683536624647/2142402696*tn^7 + 1573547024015/805
15134*tn^5, 0, 0, 0, 0, 1794626188667717/16246553778*tn^9 - 12386004853480749489
787/22812598487378700*tn^7 + 30386878785608837/80095055040*tn^5]
```

Observe that

$$L_{35} = -\frac{41683536624647}{2142402696}{\tau_n}^7 + \frac{1573547024015}{80515134}{\tau_n}^5.$$

If we want this to be of order $n^2 + n - 2 = 6^2 + 6 - 2 = 40$ then $L_{35} = 0$. The values of τ_n for which this is satisfied this are

$$\begin{aligned}
\tau_{n_1} &= 0, \\
\tau_{n_2} &= \frac{6}{958721342366881}\sqrt{25646082957398216075410002655}, \\
\tau_{n_3} &= -\frac{6}{958721342366881}\sqrt{25646082957398216075410002655},
\end{aligned}$$

We also have

$$\begin{aligned}
L_{40} = \frac{1794626188667717}{16246553778}{\tau_n}^9 - \frac{12386004853480749489787}{22812598487378700}{\tau_n}^7 + \\
+ \frac{30386878785608837}{80095055040}{\tau_n}^5.
\end{aligned}$$

Now we can check

$$\begin{aligned}
L_{40}(\tau_n = \tau_{n_1}) &= 0, \\
L_{40}(\tau_n = \tau_{n_2}) &\approx -55149 \neq 0, \\
L_{40}(\tau_n = \tau_{n_3}) &\approx 55149 \neq 0.
\end{aligned}$$

The results have been rounded because fractions with a lot of irrelevant digits have been obtained, and our only purpose is whether it is zero or not. Thus, we have checked what the article states, since we have found $\tau_n = \tau_{n_2}$ and $\tau_n = \tau_{n_3}$ such that $L_{n^2+n-2} = L_{40} \neq 0$, so there is a weak focus of order 40 at the origin.

4.3 Implementation of the interpolation technique

The interpolation technique developed in Section 3.2 as a method to find the Lyapunov quantities has also been implemented in `lyapunov_quantities.gp`. Before explaining the various functions which implement this technique, a notation for the monomials will be introduced. In order to operate with monomials in a simple and clear way, each monomial is represented in the program as a matrix. For a monomial with q factors (according to Definition 3.3), a $q \times 4$ matrix is considered, where each row corresponds to a monomial factor. Now using the notation in Definition 3.3, the first two columns contain the subindices k and l respectively of the corresponding factor. The third column contains value 1 if the factor is non-conjugated, and -1 if the factor is conjugated. Finally, the last column contains the degree $m_{k,l}$ or $n_{k,l}$ of the factor. For example, monomial $r_{20}r_{21}^3\overline{r}_{03}^2$ would be matricially represented as

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 3 & -1 & 2 \end{pmatrix}.$$

This notation will be used in the program to store and handle monomials.

Now the code for the implementation of the interpolation technique consists of the following functions:

- Function ***deg(r)***. This routine computes the degree of a monomial whose matrix representation is r according to definition (27).
- Function ***qdeg(r)***. This routine computes the quasi degree of a monomial whose matrix representation is r according to definition (28).
- Function ***weight(r)***.This routine computes the weight of a monomial whose matrix representation is r according to definition (29).
- Function ***check_element(a,b)***. This function compares two monomial matrices a and b. Its objective is to detect two things:
  - Whether two matrices represent the same monomial, i.e. they only differ in the order of their rows.
  - Whether two matrices represent conjugate monomials, i.e. the signs of the values in the third column are all opposite.

  The function returns 1 if a and b are equal or conjugate and 0 in any other case.
- Function ***check_list(r,L)***. This function is applied before adding a new monomial r to a list L in order to check whether this monomial or its conjugate is

already on the list, with the purpose of not adding it if this happens. This is done by calling the function *check_element* for each element of list L to compare it with r. The reason why we do not add a monomial r if its conjugate is already in list L is that, due to the Lyapunov quantities structure, if we have a monomial we do not need its conjugate (this fact is justified by expression (30)). The function returns 1 if r or its conjugate is in list L and 0 in other case. This routine is called by other functions when creating the list of monomials which appear in the expression of the Lyapunov quantity.

- Function ***arrange_monomial(r)***. Given a monomial matrix r, this function returns a matrix for the same monomial but grouping the factors which are equal assigning to them the corresponding degree. This is a support function which is called by other functions in the code.

- Function ***theoremA(monomials)***. According to Theorem 3.7, for those monomials with even degree only the imaginary part is a term of the Lyapunov quantity. Observe that a monomial which is formed only by a certain number of factors and their conjugates has always even degree and it is a real number, since the product of a number and its conjugate is a real value. Therefore, Theorem 3.7 allows to discard monomials of this kind because they will not be part of the Lyapunov quantity. Function *theoremA* receives a list *monomials* of monomials and returns the same list from which those monomials which are formed only by a certain number of factors and their conjugates have been removed.

- Function ***create_monomials(par,M)***. This routine creates the list of monomials which, according to Theorem 3.6, can be a term of the M-th Lyapunov quantity for a differential equation with the coefficients in argument *par* (*par* is a list of row vectors $[i, j]$ which contains the corresponding subindices). This means that the function selects all those monomials with quasi degree $2M$ and weight zero. After doing this, function *theoremA* is applied on the resulting list to remove some monomials which do not appear in the expression of the Lyapunov quantity.

- Function ***convert_monomial(r)***. Given a monomial matrix r, this routine returns the expression of the corresponding monomial, denoting the coefficient of $z^i w^j$ as r_{ij} and its conjugate as Cr_{ij}.

- Function ***conjugate_monomial(r)***. It returns the matrix corresponding to the conjugate of the monomial represented by matrix r by changing the sign to all the third column.

- Function ***generate_equation(r)***. This function generates a complex differential equation of the form $iz + Z(z, w)$, where $Z(z, w)$ is formed by the terms corresponding to the factors of monomial r with randomly selected complex coefficients. This routine will be used to generate random equations in order to perform the interpolation which allows to find the Lyapunov quantity.

- Function ***interpolation(par,M,syst)***. This is the main function of the implementation of the interpolation technique. The routine computes the M-th Lyapunov quantity of a differential equation described by *par*, which is a list of subindices. The argument *syst* is the list or vector of differential equations which are used to perform the interpolation. If this argument is given 0, the equations are randomly generated, and if not enough equations are given then the rest of equations are also randomly generated.

  The routine works as follows. First a list of suitable monomials are created by means of function *create_monomials*, and their conjugates are also added. Then, if argument *syst* is 0 a set of differential equations is randomly generated with function *generate_equation(r)*, and if *syst* is an insufficient set of differential equations it is also completed with randomly generated equations. The M-th Lyapunov quantities of these equations are computed using the Lyapunov method implementation of the previous section. Finally, a linear system is constructed with this information, and solved if possible to obtain the Lyapunov quantity via interpolation. This system can require a high use of memory, so an *allocatemem* function is executed at the beginning of the code to increase the stack size to 4096000000.

  A loop controls that the generated system can be solved by means of the system matrix rank. If the generated system cannot be solved then a new random system is created, and if after 1000 system generations a suitable system has not been created then an error message is given. The code returns either the M-th Lyapunov quantity if the resolution is successful or an error messsage otherwise.

4.3.1 Verification of the code with an example

Example 1

Let us consider the differential equation, also extracted from [14],

$$\dot{z} = iz + w^{n-1} + z^n.$$

According to Theorem 1.3 from reference [14], the origin of this equation is an stable (resp. unstable) weak focus of order $(n-1)^2$ when n is even (resp. odd), for $3 \leq n \leq 100$. Let us check this for $n = 3, 4, 5, 6$ with the implemented PARI code. To apply the interpolation technique we can write

```
? Ln=vector((3-1)^2);
? for(q=1,(3-1)^2,Ln[q]=substvec(interpolation([[3,0],[0,3-1]],q,0),[r30,Cr30,r02,Cr02],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 1]
? Ln=vector((4-1)^2);
? for(q=1,(4-1)^2,Ln[q]=substvec(interpolation([[4,0],[0,4-1]],q,0),[r40,Cr40,r03,Cr03],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 0, 0, 0, 0, 0, -91/40]
? Ln=vector((5-1)^2);
? for(q=1,(5-1)^2,Ln[q]=substvec(interpolation([[5,0],[0,5-1]],q,0),[r50,Cr50,r04,Cr04],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 167501/17325]
? Ln=vector((6-1)^2);
? for(q=1,(6-1)^2,Ln[q]=substvec(interpolation([[6,0],[0,6-1]],q,0),[r60,Cr60,r05,Cr05],[1,1,1,1]));
? print(Ln);
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3991590097/62741952]
```

Now in order to compare with the Lyapunov method, we create the following loop and obtain the same results:

```
? for(n=3,6,lyapunov_method(I*z+w^(n-1)+z^n,(n-1)^2);print(L);)
[0, 0, 0, 1]
[0, 0, 0, 0, 0, 0, 0, 0, -91/40]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 167501/17325]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3991590097/62741952]
```

We have stated that for a certain n there is a focus of order $(n-1)^2$. This means that the first Lyapunov quantity different from 0 is the $(n-1)^2$-th, and this is why for each n the first $(n-1)^2$ first quantities have been computed. Observe that we obtain what we expected, as we see that for each n we have $L_k = 0$ for $k < (n-1)^2$ and $L_{(n-1)^2} \neq 0$. Furthermore, as the paper predicts, we observe in our results that for odd n ($n = 3, 5$) then $L_{(n-1)^2} > 0$, this is an unstable weak focus, and for even n ($n = 4, 6$) then $L_{(n-1)^2} < 0$, this is a stable weak focus.

4.4 An approach to the parallelization of the problem

The interpolation technique has a great computational advantage with respect to the Lyapunov method: it can be parallelized. Performing a complete parallelization of the problem is out of the scope of this project, I will work on it during my PhD. However, I find interesting to introduce a first step towards this parallelization and analyzing the efficiency improvement that can be achieved.

The high computational cost of the methods developed in this project justify the necessity of implementing a parallelization technique on them. As previously explained, the interpolation technique intoduced above allows to split a complicated and slow problem into several simple and fast problems. Particularly, instead of applying the Lyapunov method to the original equation it can be applied to several simpler equations. Furthermore, this approach has the advantage that the many calls to this function can be parallelized, which reduces execution times and increases efficiency. Therefore, the part of the interpolation technique which can be potentially parallelized are the many executions of the Lyapunov method.

For the parallelization I will connect to Antz, the computing servers in the Mathematics Department of UAB (Universitat Autònoma de Barcelona). The reason for using these servers is that my future PhD Thesis will be developed in the Mathematics Department of UAB, and I have been signed up as a user in Antz so as to have access to the computer infrastructure. The connection to the Antz servers is carried out using WinSCP and Putty.

The software used to perform the parallelization will be PBala[4], a distributed execution software for Antz developed by Oscar Saleta, who is a research support specialist in the Mathematics Department of UAB. PBala is a parallellization interface for single threaded scripts, which allows to distribute executions in Parallel Virtual Machine enabled clusters using single program multiple data paradigm. This interface lets the user execute a same script/program over multiple input data in several CPUs located at the Antz computing servers. It supports memory management so nodes do not run out of RAM due to too many processes being started in the same node. It also reports resource usage data after execution.

PBala allows the parallelization of codes in many different languages: Maple, C, Python, Pari/GP, Sage and Octave. In our case we will use the Pari/GP option, as the Lyapunov quantities computation code is written in this language. For the parallelization the following Pari/GP code will be used

```
\r lyapunov_method.gp
print(taskId);
for(i=1,length(taskArgs),lyapunov_method(taskArgs[i],30);print(L););
```

This code is saved in a Pari/GP file `solve_lyapunov_system.gp`. The first line loads the functions from the Pari/GP file `lyapunov_quantities.gp` for the Lyapunov quantities computation. The parallelization divides the problem in tasks, and line 2 in the code prints the identification number for the current task, which allows to keep track of the process. The third line performs the Lyapunov method (by means of function `lyapunov_method` from `lyapunov_quantities.gp`) for computing and printing the 30 first Lyapunov quantities of the differential equation which are passed as arguments.

Let us see now with a bit more of detail how to perform this parallelization. First, we need to have a data file which contains the data to be passed as arguments in the parallel code. Each row is a single execution and has each value separated by a comma, line format being "tasknumber,arg1,arg2,...,argN". The first value in each row must be a number and it is the task identification number, which will be stored in `taskId`. The rest of each row are the arguments to be passed to the code, which are stored in `taskArgs`. For row `i`, the arguments are contained in `taskArgs[i]`. Thus, when we perform the instruction `lyapunov_solve(taskArgs[i],30)`, we are

[4] https://github.com/oscarsaleta/PBala

finding the first 30 Lyapunov quantities for the nonlinearity stored in row `i` from the data file.

It is also needed to create a node file, which contains the number of processes to be assigned to each node. In Antz there are eight nodes, called a01, a02, ..., a08. Then, the node file line format is "nodename number_of_processes".

With this, the instruction to execute the parallel code will be

```
time ./PBala -eh 3 solve_lyapunov_system.gp datafile.txt nodefile.txt
  results
```

The options `-eh` tell PBala that we want to generate error files (in case something goes wrong) and a slave file (that tells us which node has performed which execution). Number `3` in this line tells PBala that the language to use is Pari/GP, and each programming language is represented by a different number. Then the program, the data file and the node file must be indicated. Finally, `results` is the output directory for storing the results. The option `time` at the beginning makes the execution show the runtime.

As we have already stated, we will not perform a general parallelization of the problem, but we will present a particular example of this and how execution times are reduced. To carry out this analysis we will consider the example of [11] of computing some Lyapunov quantities of the differential equation

$$\dot{z} = iz + A\,z^2 + B\,zw + C\,w^2 + D\,z^3 + E\,z^2w + F\,zw^2 + G\,w^3 + H\,z^4 + I\,z^3w \\ + J\,z^2w^2 + K\,zw^3 + L\,w^4 + M\,z^5 + N\,z^4w + O\,z^3w^2 + P\,z^2w^3 + Q\,zw^4 + R\,w^5.$$

To simplify the problem, we will use the interpolation technique and consider a set of simpler nonlinearities to apply the Lyapunov method on them. According to [11], a suitable set of 28 simple polynomial nonlinearities $Z(z,w)$ which allow to solve this problem is the following:

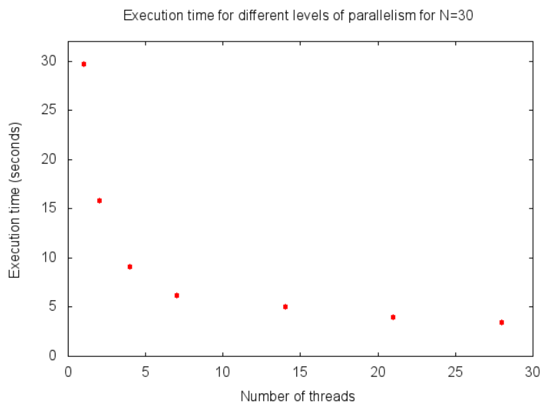
z^3w^2	w^2+izw^3	$z^2+w^2+zw^2$	$z^2+izw+(1+i)z^2w$
z^3-izw^2	z^2+zw^2	$zw+w^2+zw^2$	$z^2+(1+i)zw+(1+i)z^2w$
z^2-iz^3w	$zw-z^3$	$zw+w^2+z^3$	$(1+i)z^2+zw+(1+i)z^2w$
$z^2+iz^2w^2$	$zw+zw^2$	$z^2+w^2+w^3$	$z^2+(1+i)zw+z^2w$
$zw-iz^3w$	$z^2+zw+zw^2$	$zw+w^2+w^3$	$z^2-izw+w^2-z^2w$
$zw+iz^2w^2$	z^2+zw-z^3	iz^2+w^2	$iz^2+zw+w^2+z^2w$
w^2+iz^4	$z^2+w^2-z^3$	$zw+iw^2$	$iz^2+izw+w^2$

These 28 polynomials are then included in the data file (as shown in Appendix B) to be taken as the arguments of the Lyapunov method.

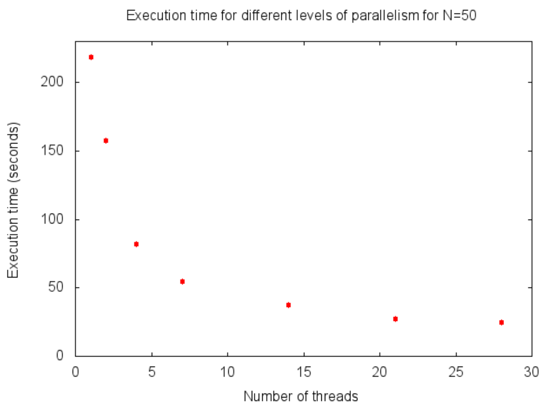
Our aim now is to compute the first N Lyapunov quantities for the differential equations (3), taking as $Z(z,w)$ the previous 28 polynomials. This will be done with different levels of parallelization[5] and their execution times will be compared. In particular, we have taken as numbers of threads 1, 2, 4, 7, 14, 21 and 28, and this information is given to PBala by means of the node file. Then the Lyapunov quantities computation has been performed taking $N = 30$ and $N = 50$. The execution time results are shown in Table 1, and graphically in Figure 4.

Table 1: Execution times for different levels of parallelization

Number of threads	Execution time for $N = 30$ (seconds)	Execution time for $N = 50$ (seconds)
1	29.741	218.823
2	15.798	157.654
4	9.096	81.911
7	6.146	54.347
14	5.005	37.431
21	4.001	27.603
28	3.409	24.633



(a) $N = 30$



(b) $N = 50$

Figure 4: Execution time as a function of the level of paralellism for the computation of the first 30 and 50 Lyapunov quantities.

We can make some observations on the plots in Figure 4. The most obvious and expectable observation is the fact that, as the level of paralellism increases, the execution time is reduced, because the execution is distributed in a greater number of threads and this makes it faster. Furthermore, as Figures 4a and 4b have the same shape, we can deduce that the relative improvement by parallelization is always the

[5]The level of parallelization refers to the number of threads, this is, the number of parallelized tasks that the system can perform at the same time.

same independently on the problem size N.

The difference between both N is that, as can be seen in Table 1, when passing from $N = 30$ to $N = 50$, the execution times approximately increase at least 7 times. This shows that the Lyapunov constants computation complexity does not increse linearly with the problem size. Taking into account the Lyapunov method algorithm this is what we expect, since for computing the first N constants it is necessary to solve linear equations systems whose matrices have order $N \times N$, which quickly increases the computational cost (at least with order N^2). For this reason, as the problem size is increased parallelization becomes more useful, and even necessary.

Moreover, we can see in Figure 4 that both plots become flatter as the number of threads increases, which indicates that the improvement by parallelization is lower for high levels of parallelism. This fact corroborates the so-called Amdahl's Law, a principle which states that there is a speedup limitation that makes that, for a certain number of threads, increasing even more the number of threads is not worth.

This example justifies the need of parallelization and the great benefit it provides in the computation of the Lyapunov quantities and, more generally, in the resolution of the center-focus and ciclicity problems. The reason is that the systems to solve quickly become enormous and they require a high computational efficiency. This is just what we have seen with the enormous improvement in the computation time of Lyapunov quantities for big problem sizes: comparing the execution time between 1 thread (no parallelization) and 28 threads (maximum parallelization) in Table 1, we can see a reduction of approximately 90% in computation time.

5 Resolution for certain polynomial families

5.1 A cubic polynomial family

In this section we will show how to solve the center-focus and ciclicity problems for the cubic polynomial family

$$\begin{cases} \dot{x} = -y + a_2x^2 + a_3x^3, \\ \dot{y} = x + b_2x^2 + b_3x^3. \end{cases} \tag{32}$$

Notice that applying the time change $t \to -t$ these equations have the form of a Liénard system (13). Using the software developed during the previous section, we can compute the first Lyapunov quantity of this system

$$L_1 = \frac{3}{8}a_3 - \frac{1}{4}a_2b_2.$$

Now the second Lyapunov quantity under condition $L_1 = 0$ is

$$L_2 = \frac{5}{24}a_2b_2b_3.$$

If we impose restriction $L_1 = L_2 = 0$ and compute the following Lyapunov quantities we obtain $L_3 = 0$, $L_4 = 0$, $L_5 = 0$. This makes us wonder whether $L_k = 0$ for every $k \geq 3$. We will deal with this problem in the following subsections.

5.1.1 Center-focus problem

Let us first analyze for which parameters our system has a center at the origin. This happens if and only if $L_k = 0$ for every $k \geq 1$. Thus, we have to study the cases $L_1 = 0$ and $L_2 = 0$ and find methods to prove that in these situations the origin is actually a center. These are the only cases in which the origin can be a center, since if $L_1 \neq 0$ or $L_2 \neq 0$ we know that the origin must be a focus. Then we are interested in solving the system

$$\begin{cases} L_1 = 0, \\ L_2 = 0. \end{cases} \iff \begin{cases} 3a_3 - 2a_2b_2 = 0, \\ a_2b_2b_3 = 0. \end{cases}$$

This system has three possible solutions:

(i) $a_2 = a_3 = 0$; b_2 and b_3 free parameters;

(ii) $b_2 = a_3 = 0$; a_2 and b_3 free parameters;

(iii) $b_3 = 0$; $a_3 = \frac{2}{3}a_2b_2$; a_2 and b_2 free parameters.

We have then three cases to decide whether the origin is a center.

- Case (i). The system has the form

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + b_2x^2 + b_3x^3. \end{cases}$$

Applying a time and coordinate change $(x, y, t) \to (x, -y, -t)$ we see that the system remains invariant. Thus, as we saw in Subsection 2.3.3, in this case the orbits close on themselves and form periodic orbits, so the origin of the system is a center.

- Case (ii). The system has the form

$$\begin{cases} \dot{x} = -y + a_2x^2, \\ \dot{y} = x + b_3x^3. \end{cases}$$

With a time and coordinate change $(x, y, t) \to (-x, y, -t)$ we observe that this system also remains invariant. Thus, with an analogous reasoning to (i), this system has a center at the origin.

- Case (iii). In this last case the system is

$$\begin{cases} \dot{x} = -y + a_2x^2 + \frac{2}{3}a_2b_2x^3, \\ \dot{y} = x + b_2x^2. \end{cases}$$

Here we cannot use time and coordinate changes as we did in the previous cases to obtain symmetries, so we need to find other techniques. Applying a time change $t \to -t$ the equations can be rewritten as

$$\begin{cases} \dot{x} = y - a_2x^2 - \frac{2}{3}a_2b_2x^3, \\ \dot{y} = -x - b_2x^2. \end{cases}$$

Observe that this system has exactly the form of a Liénard system (13) with

$$F(x) = a_2x^2 + \frac{2}{3}a_2b_2x^3, \qquad g(x) = x + b_2x^2.$$

Let us now compute the primitive $G(x)$ of $g(x)$:

$$G(x) = \int_0^x g(s)\, ds = \int_0^x (s + b_2s^2)\, ds = \frac{1}{2}x^2 + \frac{1}{3}b_2x^3.$$

Comparing $F(x)$ and $G(x)$ we see that

$$F(x) = a_2x^2 + \frac{2}{3}a_2b_2x^3 = 2a_2\left(\frac{1}{2}x^2 + \frac{1}{3}b_2x^3\right) = 2a_2G(x) =: \Phi(G(x)),$$

where we define function $\Phi(x) = 2a_2x$. This function is clearly analytic and satisfies $\Phi(0) = 0$, so we can apply Theorem 2.11 to prove that in this case the origin of this system is also a center.

These are the three only possibilities for the origin of being a center. Thus, the conditions we have found here which make L_1 and L_2 vanish are the center conditions of system (32), and the center-focus problem for this system is solved by the following theorem:

Theorem 5.1. *The system of differential equations (32) has a center at the origin if and only if at least one of the following conditions is satisfied:*

(i) $a_2 = a_3 = 0$.

(ii) $b_2 = a_3 = 0$.

(iii) $b_3 = 0$ *and* $a_3 = \frac{2}{3}a_2 b_2$.

A complex coordinates version of this theorem can be found in [24].

5.1.2 Ciclicity problem

We have seen that, for system (32), if $L_1 = L_2 = 0$ then the system has a center at the origin and then $L_k = 0$ for every $k \geq 3$. This makes us wonder if every L_k with $k \geq 3$ belongs to the ideal generated by L_1 and L_2. Choosing $m = 2$ on (11), this would mean that

$$\langle L_1, L_2, L_3, \ldots \rangle = \langle L_1, L_2 \rangle. \tag{33}$$

We will show that this equality holds, and this implies that the system has at most two limit cycles which bifurcate from the origin.

Assume that $L_1 \neq 0$, then there clearly exists λ_1 such that

$$\Pi(\rho, \lambda_1) - \rho = L_1 \rho^3 + \cdots .$$

Now slightly perturb the system taking $\lambda \sim \lambda_1$. We want to compute the zeros of the distance function $f(\rho, \lambda \sim \lambda_1)$. As we saw in Section 2.4, applying the Weierstrass Preparation Theorem this problem reduces to finding the zeros of polynomial $\rho^3 + s_2(\lambda)\rho^2 + s_1(\lambda)\rho$. We also justified that, as in this case $k = 3$, then the system has at most $(k-1)/2 = (3-1)/2 = 1$ limit cycle.

Assume now that $L_1 = 0$ and $L_2 \neq 0$, then there exsits λ_2 such that

$$\Pi(\rho, \lambda_2) - \rho = L_2 \rho^5 + \cdots .$$

Analogously, applying the Weierstrass Preparation Theorem, we know that finding the zeros of $f(\rho, \lambda \sim \lambda_2)$ is equivalent to finding the zeros of $\rho^5 + t_4(\lambda)\rho^4 + t_3(\lambda)\rho^3 + t_2(\lambda)\rho^2 + t_1(\lambda)\rho$. In this case, as $k = 5$ we can conclude that the system has at most $(k-1)/2 = (5-1)/2 = 2$ limit cycles.

The problem gets a bit harder when $L_1 = L_2 = 0$. We saw in the previous subsection that in this case the origin is a center, so

$$\Pi(\rho, \lambda_0) - \rho \equiv 0.$$

Here the Weierstrass Preparation Theorem cannot be applied to study what happens when perturbing λ_0 because the function is identically zero and then the conditions of the theorem are not satisfied. Therefore, in this case other techniques must be used to study function $f(\rho, \lambda) = \Pi(\rho, \lambda) - \rho$ for $\lambda \sim \lambda_0$.

Let J be $\langle L_1, L_2 \rangle$. Using instruction `IsRadical` from the package `PolynomialIdeals` in Maple we can easily check that this ideal J is radical. Now let $V(J)$ be the set of zeros of ideal J –recall that it is the set of parameters for which the elements of $J = \langle L_1, L_2 \rangle$ vanish. Consider now the ideal of set $V(J)$, denoted as $\mathcal{I}(V(J))$.

As we have seen that if $L_1 = L_2 = 0$ then the system has a center and $L_k = 0$ for every $k \geq 3$, we have that the set of parameters which make L_1 and L_2 vanish will automatically make $L_k = 0$ vanish for every $k \geq 3$. This implies that $L_k \in \mathcal{I}(V(J))$ for every $k \geq 3$. Therefore, due to Corollary 2.7 of the Hilbert Zeros Theorem, we also have that $L_k \in J = \langle L_1, L_2 \rangle$ for every $k \geq 3$, and this finally proves that $\langle L_1, L_2, L_3, \ldots \rangle = \langle L_1, L_2 \rangle$. As a consequence, if $k \geq 3$ then

$$L_k = r_k(x) L_1 + s_k(x) L_2,$$

for certain polynomials $r_k(x)$ and $s_k(x)$. We have then that the distance function for a perturbation $\lambda \sim \lambda_0$ is

$$\begin{aligned} f(\rho, \lambda) =& L_1 \rho^3 + W_4 \rho^4 + L_2 \rho^5 + W_6 \rho^6 + L_3 \rho^7 + \cdots = \\ =& L_1 \rho^3 \left(1 + a_1(\rho, \lambda)\rho + a_2(\rho, \lambda)\rho^2 + \cdots\right) + \\ & + L_2 \rho^5 \left(1 + b_1(\rho, \lambda)\rho + b_2(\rho, \lambda)\rho^2 + \cdots\right). \end{aligned}$$

Notice that, for every $j \geq 2$, we have $W_{2j} = 0$ when the Lyapunov quantity $W_{2j-1} = L_{j-1}$ vanishes, and for this reason the common factors $L_1 \rho^3$ and $L_2 \rho^5$ can be extracted.

Finally we can now apply the Weierstrass Preparation Theorem for each of the addends in the previous expression, and we obtain that

$$f(\rho, \lambda) = a(\rho, \lambda)\rho^3 + b(\rho, \lambda)\rho^5.$$

As it is a degree $k = 5$ polynomial, function $f(\rho, \lambda)$ for $\lambda \sim \lambda_0$ has at most $(5 - 1)/2 = 2$ positive zeros, which means that at most two limit cycles appear in a neighbourhood of the origin when perturbing the system.

Notice that the fact that the polynomials introduced here have two positive zeros is due to the fact that L_1 and L_2 in (33) take an arbitrary value. Thus, the procedure we have seen is general for any system where $\langle L_1, L_2 \rangle$ is a radical ideal which satisfies (33).

5.2 Bautin's quadratic polynomials

In his article [3] from 1952, Bautin solved the problem of finding the maximum number of limit cycles for quadratic differential systems, these are systems of the form

$$\begin{cases} \dot{x} = a_{10}\,x + a_{01}\,y + a_{20}\,x^2 + a_{11}\,xy + a_{02}\,y^2, \\ \dot{y} = b_{10}\,x + b_{01}\,y + b_{20}\,x^2 + b_{11}\,xy + b_{02}\,y^2, \end{cases} \tag{34}$$

In this subsection we will explicitly solve the center-focus problem for system (34) and we will explain how the maximum number of limit cycles can be found.

5.2.1 Center-focus problem

According to [9], if system (34) has a center at the origin then a linear transformation and a time change can be applied in order to rewrite the system as

$$\begin{cases} \dot{x} = -y - bx^2 - Cxy - dy^2, \\ \dot{y} = x + ax^2 + Axy - ay^2. \end{cases} \tag{35}$$

Using the developed program the first Lyapunov quantities can be computed,

$$\begin{aligned} L_1 &= \frac{1}{8}(b+d)(2a+C), \\ L_2 &= -\frac{1}{96}C(b+d)(A-2b)(A+3b+5d), \\ L_3 &= -\frac{5}{512}C(b+d)^2(C^2+4bd+8d^2)(A-2b). \end{aligned}$$

Notice that L_2 has been computed with the condition $L_1 = 0$ and L_3 with conditions $L_1 = L_2 = 0$. If we compute some of the following Lyapunov quantities we see that they are 0, which indicates that maybe $L_k = 0$ for $k \geq 4$; we will deal with this issue later. Now let us solve the system

$$\begin{cases} L_1 = 0, \\ L_2 = 0, \\ L_3 = 0. \end{cases} \iff \begin{cases} (b+d)(2a+C) = 0, \\ C(b+d)(A-2b)(A+3b+5d) = 0, \\ C(b+d)^2(C^2+4bd+8d^2)(A-2b) = 0. \end{cases}$$

This system has four possible solutions:

(i) $A = 2b$ and $C = -2a$;

(ii) $C = a = 0$;

(iii) $d = -b$;

(iv) $C = -2a$, $A + 3b + 5d = 0$ and $a^2 + bd + 2d^2 = 0$.

Therefore we have two cases to analyze whether the origin is a center.

- Case (i). The system has the form

$$\begin{cases} \dot{x} = -y - bx^2 + 2axy - dy^2, \\ \dot{y} = x + ax^2 + 2bxy - ay^2. \end{cases}$$

  In this case the system is Hamiltonian[6], and it is easy to check that function

$$H = \frac{1}{2}(x^2 + y^2) + \frac{a}{3}x^3 + bx^2y - axy^2 + \frac{d}{3}y^3$$

  is a first integral defined near the origin. Thus, as there exists a first integral the origin must be a center.

- Case (ii). In this case the system is

$$\begin{cases} \dot{x} = -y - bx^2 - dy^2, \\ \dot{y} = x + Axy. \end{cases}$$

  It is invariant under the change $(x, y, t) \to (-x, y, -t)$, and we have seen that if this happens then there is a symmetry that proves that the origin is a center.

- Case (iii). If $a \neq 0$, there exists a certain rotation such that the new $a' = 0$. Thus, we can assume $a = 0$ (see [9] for more details). Then, if $d = -b$ and $a = 0$ the system has the form

$$\begin{cases} \dot{x} = -y - bx^2 - Cxy + by^2, \\ \dot{y} = x + Axy. \end{cases}$$

  This system has the invariant curves $f_1 = 1 + Ay = 0$ (if $A \neq 0$) with cofactor $K_1 = Ax$ and $f_2 = (1 - by)^2 + C(1 - by)x - b(A + b)x^2 = 0$ with cofactor $K_2 = -2bx - Cy$. We also compute $\text{div}(P, Q) = -2bx - Cy + Ax = K_1 + K_2$. Therefore, we have that there exist $\lambda_1 = -1$, $\lambda_2 = -1$ such that $\lambda_1 K_1 + \lambda_2 K_2 = -\text{div}(P, Q)$, and applying (ii) of Darboux Theorem 2.17, we have that $f_1^{-1} f_2^{-1}$ is an integrating factor. As the first integral associated to the integrating factor is defined near the origin, we can conclude that the origin is a center. If $A = 0$ then f_1 is not an invariant curve, but in this case the system divergence is K_2 and analogously we see that the integrating factor is f_2^{-1} and the origin is a center.

[6] A Hamiltonian system is a differential system for which there exists a function H such that $\dot{x} = -\partial H/\partial y$ and $\dot{y} = \partial H/\partial x$. Furthermore, it can be trivially checked that this function H is a first integral of the system

- Case (iv). Assume that $d \neq 0$ (if $d = 0$ we reduce to case (ii)). Then we write the system as

$$\begin{cases} \dot{x} = -y + \dfrac{a^2 + 2d^2}{d} x^2 + 2axy - dy^2, \\ \dot{y} = x + ax^2 + \dfrac{3a^2 + d^2}{d} xy - ay^2. \end{cases}$$

  The system has an invariant curve $f_1 = (a^2 + d^2)\left[(dy - ax)^2 + 2dy\right] + d^2 = 0$ with cofactor $K_1 = 2(a^2 + d^2)x/d$. If we compute the divergence of the system in this case we obtain $\frac{5}{2}K_1$, and applying (ii) of Darboux Theorem 2.17 we deduce that $f_1^{-5/2}$ is an integrating factor. As $d \neq 0$, the associated first integral is defined in a neighbourhood of the origin, and therefore the origin is a center.

We have seen some conditions under which system (35) has a center. Observe that these are the only possibilities for the system to have a center at the origin, because if they are not satisfied then either $L_1 \neq 0$, $L_2 \neq 0$ or $L_3 \neq 0$, and the origin would be a focus. Thus, we have proved that the center-focus problem in this case is solved by the following theorem, which according to [9] is known as the Kapteyn-Bautin Theorem.

Theorem 5.2. ***(Kapteyn-Bautin Theorem)*** *The differential equations system (35) has a center at the origin if and only if at least one of the following conditions is satisfied:*

(i) $A = 2b$ *and* $C = -2a$.

(ii) $C = a = 0$.

(iii) $d = -b$.

(iv) $C = -2a$, $A + 3b + 5d = 0$ *and* $a^2 + bd + 2d^2 = 0$.

5.2.2 Ciclicity problem

We have just proved that, for the aforementioned quadratic family, if $L_1 = L_2 = L_3 = 0$ then the system has a center at the origin and therefore $L_k = 0$ for every $k \geq 4$. Recall what we saw about ciclicity in Subsection 2.4. We have that if $L_1 \neq 0$ then at most one limit cycle will appear when perturbing the system. If $L_1 = 0$ and $L_2 \neq 0$, it will have at most two limit cycles; if $L_1 = L_2 = 0$ and $L_3 \neq 0$ there will be at most three limit cycles. We can deduce this because the Preparation Weierstrass Theorem can be applied.

As in the case $L_1 = L_2 = L_3 = 0$ the system has a center, then under these conditions

$$\Pi(\rho, \lambda_0) - \rho \equiv 0,$$

so the Preparation Weierstrass Theorem cannot be applied.

We saw in the example of the cubic polynomials that the fact that the ideal generated by the two first Lyapunov quantities was radical already solved the problem. In the current case of the quadratic polynomials, again using Maple we see that ideal $\langle L_1, L_2, L_3 \rangle$ is not radical, so we cannot use the techniques we used for system (32) of cubic polynomials and the problem becomes much more complicated. Despite this, Bautin proved in his article [3] that the maximum number of limit cycles which can appear near the origin when perturbing the system is three. In this article, Bautin explicitly deduced how the Lyapunov quantities L_k for $k \geq 4$ are a linear combination of quantities L_1, L_2 and L_3, so proceeding as in the cubic example we can conclude that the maximum number of limit cycles which can appear when perturbing the system is three.

We will not go deeper in the problem of the ciclicity for this quadratic family; it has only been introduced here as an example of how the problem becomes much harder when the ideal generated by the Lyapunov quantities is not radical. In this case, the resolution of the ciclicity problem is not automatic and other techniques are required.

5.3 Homogeneous nonlinearities

5.3.1 A conjecture on the number of Lyapunov quantites

In this subsection, a study on a polynomial family with homogeneous nonlinearities will be carried out. This analysis will be based on the following conjecture, extracted from [12],

Conjecture 5.3. *Given a differential equation defined by homogeneous polynomial nonlinearities of degree n, the minimum number of Lyapunov quantities which are necessary to solve the center-focus problem is $2n-1$.*

This conjecture has already been proved for degree 2 and 3 polynomial nonlinearities (see [20], [24]). In the following part we will perform the computation of the Lyapunov quantities of a family with degree 5 homogeneous nonlinearities in order to check that the conjecture also holds in this case.

5.3.2 A fifth degree family

Here we will study the following polynomial family with fifth degree homogeneous nonlinearities,

$$\dot{z} = iz + a_{50}z^5 + a_{32}z^3w^2 + a_{14}zw^4. \tag{36}$$

Using the previously developed program, the Lyapunov quantities of this differential equation can be found. The first 20 Lyapunov quantities have been computed and they are shown in Appendix C. We can see there that those quantities L_k with odd $k \leq 20$ are identically zero, while those with even $k \leq 20$ are not. Now we load those quantities with Maple and use the following instruction:

```
simplify(L20,[L2,L4,L6,L8,L10,L12,L14,L16,L18]);
```

A 0 result is then obtained. This means that, if the system satisfies the conditions for which $L_2 = L_4 = L_6 = L_8 = L_{10} = L_{12} = L_{14} = L_{16} = L_{18} = 0$, then L_{20} is automatically 0. Therefore, it is expectable that the first $2n-1 = 2\cdot 5-1 = 9$ non-identically zero quantities solve the center-focus problem, as Conjecture 5.3 states. This will be studied with more detail here.

Let us first find the conditions on the coefficients in equation (36) for which $L_2 = L_4 = L_6 = L_8 = L_{10} = L_{12} = L_{14} = L_{16} = L_{18} = 0$. This can be done with the following Maple instruction:

```
solve({L2,L4,L6,L8,L10,L12,L14,L16,L18})
```

This gives three possible solutions:

(i) $\overline{a_{32}} = -a_{32}$, $a_{14} = 0$ and $\overline{a_{50}} = 0$.

(ii) $\overline{a_{32}} = -a_{32}$, $a_{50} = 0$ and $\overline{a_{50}} = 0$.

(iii) $\overline{a_{32}} = -a_{32}$ and $\overline{a_{14}a_{50}} = a_{14}a_{50}$.

Now we will prove that under these conditions the system has a center at the origin, which will complete the resolution of the center-focus problem for this equation.

- Case (i). Let us define $a_{kl} := b_{kl} + i\, c_{kl}$ for any pair of subindices $k, l \in \mathbb{N} \cup \{0\}$. Observe that the condition $\overline{a_{32}} = -a_{32}$ implies that

  $$-b_{32} - i\, c_{32} = -a_{32} = \overline{a_{32}} = b_{32} - i\, c_{32} \Rightarrow -b_{32} = b_{32} \Rightarrow b_{32} = 0 \Rightarrow a_{32} = i\, c_{32},$$

  for $c_{32} \in \mathbb{R}$. Furthermore, as $\overline{a_{50}} = b_{50} - i\, c_{50} = 0$, we immediately have that $a_{50} = b_{50} + i\, c_{50} = 0$. Therefore, in this case equation (36) can be written as

  $$\dot{z} = iz + i\, c_{32} z^3 w^2 = i(z + c_{32} z^3 w^2), \tag{37}$$

  for $c_{32} \in \mathbb{R}$. This system can be written in real coordinates as

  $$\begin{cases} \dot{x} = -y - c_{32}x^4 y - 2c_{32}x^2 y^3 - c_{32}y^5, \\ \dot{y} = x + c_{32}x^5 + 2c_{32}x^3 y^2 + c_{32}xy^4. \end{cases}$$

  Now we can check that this system is invariant under the change $(x, y, t) \to (-x, y, -t)$, and as we have seen this symmetry proves that the origin is a center.

There is another alternative to prove this. As equation (37) has the form (15), we could also apply (*iii*) of Theorem 2.18 to show that the origin is a center. Using the notation of the theorem, in our case we have $k = l = 0$, $m = 3$, $n = 2$ $(k+l = 0 \leq 3+2 = m+n)$, $\alpha = 0-0-1 = -1$ and $\beta = 3-2-1 = 0$. Then $A = 0$ and $B = i\,c_{32}$, so effectively

$$A = -\bar{A}\,e^{i\alpha\varphi} = 0,$$
$$i\,c_{32} = B = -\bar{B}\,e^{i\beta\varphi} = -\overline{i\,c_{32}}\,e^{i\cdot 0\cdot\varphi} = -\overline{i\,c_{32}} = -(-i\,c_{32}) = i\,c_{32}.$$

Thus, condition (*iii*) of the theorem is satisfied and therefore the origin is a center.

- Case (ii). As we have seen, condition $\overline{a_{32}} = -a_{32}$ implies that $a_{32} = i\,c_{32}$ for $c_{32} \in \mathbb{R}$. Moreover, as $a_{50} = 0$, in this case the system has the form

$$\dot{z} = iz + i\,c_{32}z^3w^2 + a_{14}zw^4.$$

Again, this system has the form (15), and Theorem 2.18 can be applied. Using the notation of the theorem, $k = 3$, $l = 2$, $m = 1$, $n = 4$ $(k+l = 3+2 \leq 1+4 = m+n)$, $\alpha = 3-2-1 = 0$ and $\beta = 1-4-1 = -4$. Then $A = i\,c_{32}$ and $B = a_{14}$, so

$$i\,c_{32} = A = -\bar{A}\,e^{i\alpha\varphi} = -\overline{i\,c_{32}}\,e^{i\cdot 0\cdot\varphi} = -\overline{i\,c_{32}} = -(-i\,c_{32}) = i\,c_{32},$$
$$a_{14} = B = -\bar{B}\,e^{i\beta\varphi} = -\overline{a_{14}}\,e^{-4i\varphi}.$$

Choosing an appropriate φ such that the second equality holds we finally obtain that (*iii*) of Theorem 2.18 is satisfied, and therefore the origin is a center.

- Case (iii). The first condition $\overline{a_{32}} = -a_{32}$ implies that $a_{32} = i\,c_{32}$ for $c_{32} \in \mathbb{R}$, so in this case the system has the form

$$\dot{z} = iz + a_{50}z^5 + i\,c_{32}z^3w^2 + a_{14}zw^4. \tag{38}$$

Conjugating this equation we obtain

$$\dot{w} = -iw + \overline{a_{50}}w^5 - i\,c_{32}z^2w^3 + \overline{a_{14}}z^4w. \tag{39}$$

In this case it will be useful to express the system in polar coordinates (r, φ) instead of (z, w). To do this we consider $z = r\,e^{i\varphi}$ and $w = r\,e^{-i\varphi}$. Now if we derive,

$$\dot{z} = \frac{dz}{dr}\dot{r} + \frac{dz}{d\varphi}\dot{\varphi} = e^{i\varphi}\dot{r} + ri\,e^{i\varphi}\dot{\varphi} = e^{i\varphi}\dot{r} + iz\dot{\varphi}, \tag{40}$$

$$\dot{w} = \frac{dw}{dr}\dot{r} + \frac{dw}{d\varphi}\dot{\varphi} = e^{-i\varphi}\dot{r} + r(-i)\,e^{-i\varphi}\dot{\varphi} = e^{-i\varphi}\dot{r} - iw\dot{\varphi}. \tag{41}$$

First we are interested in finding an expression for $\dot{r}$. This can be done by substituting (40) and (41) in $\dot{z}w + \dot{w}z$ as follows,

$$\begin{aligned}
\dot{z}w + \dot{w}z &= \left(e^{i\varphi}\dot{r} + iz\dot{\varphi}\right) w + \left(e^{-i\varphi}\dot{r} - iw\dot{\varphi}\right) z, \\
\dot{z}r\, e^{-i\varphi} + \dot{w}r\, e^{i\varphi} &= e^{i\varphi}\dot{r}r\, e^{-i\varphi} + e^{-i\varphi}\dot{r}r\, e^{i\varphi}, \\
\dot{z}\, e^{-i\varphi} + \dot{w}\, e^{i\varphi} &= \dot{r} + \dot{r}, \\
\dot{r} &= \frac{\dot{z}\, e^{-i\varphi} + \dot{w}\, e^{i\varphi}}{2}.
\end{aligned} \tag{42}$$

Substituting (38) and (39) in this expression and using polar coordinates we have

$$\begin{aligned}
\dot{r} =& \frac{\dot{z}\, e^{-i\varphi} + \dot{w}\, e^{i\varphi}}{2} \\
=& \frac{\left(iz + a_{50}z^5 + i\, c_{32}z^3w^2 + a_{14}zw^4\right)\, e^{-i\varphi}}{2} \\
&+ \frac{\left(-iw + \overline{a_{50}}w^5 - i\, c_{32}z^2w^3 + \overline{a_{14}}z^4w\right)\, e^{i\varphi}}{2} \\
=& \frac{\left(ir\, e^{i\varphi} + a_{50}r^5\, e^{5i\varphi} + i\, c_{32}r^3\, e^{3i\varphi}r^2\, e^{-2i\varphi} + a_{14}r\, e^{i\varphi}r^4\, e^{-4i\varphi}\right)\, e^{-i\varphi}}{2} \\
&+ \frac{\left(-ir\, e^{-i\varphi} + \overline{a_{50}}r^5\, e^{-5i\varphi} - i\, c_{32}r^2\, e^{2i\varphi}r^3\, e^{-3i\varphi} + \overline{a_{14}}r^4\, e^{4i\varphi}r\, e^{-i\varphi}\right)\, e^{i\varphi}}{2} \\
=& \frac{ir + a_{50}r^5\, e^{4i\varphi} + i\, c_{32}r^5 + a_{14}r^5\, e^{-4i\varphi} - ir + \overline{a_{50}}r^5\, e^{-4i\varphi} - i\, c_{32}r^5 + \overline{a_{14}}r^5\, e^{4i\varphi}}{2} \\
=& r^5\, \frac{\left(a_{50}\, e^{4i\varphi} + a_{14}\, e^{-4i\varphi}\right) + \left(\overline{a_{50}}\, e^{-4i\varphi} + \overline{a_{14}}\, e^{4i\varphi}\right)}{2} \\
=& r^5\, \mathrm{Re}\left(\mathrm{a}_{50}\, \mathrm{e}^{4\mathrm{i}\varphi} + \mathrm{a}_{14}\, \mathrm{e}^{-4\mathrm{i}\varphi}\right).
\end{aligned}$$

Using notation $a_{kl} = b_{kl} + i\, c_{kl}$ and Euler's fomula[7] this can be rewritten as

$$\dot{r} = r^5 \left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi) + b_{14}\cos(4\varphi) + c_{14}\sin(4\varphi)\right). \tag{43}$$

Let us now deduce the expression for the derivative of the angular component $\dot{\varphi}$. To do this we subtitute (40) and (41) in $\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi}$ as follows,

$$\begin{aligned}
\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi} &= \left(e^{i\varphi}\dot{r} + iz\dot{\varphi}\right)\, e^{-i\varphi} - \left(e^{-i\varphi}\dot{r} - iw\dot{\varphi}\right)\, e^{i\varphi}, \\
\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi} &= \dot{r} + iz\dot{\varphi}\, e^{-i\varphi} - \dot{r} + iw\dot{\varphi}\, e^{i\varphi}, \\
\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi} &= ir\, e^{i\varphi}\dot{\varphi}\, e^{-i\varphi} + ir\, e^{-i\varphi}\dot{\varphi}\, e^{i\varphi}, \\
\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi} &= ir\dot{\varphi} + ir\dot{\varphi}, \\
\dot{\varphi} &= \frac{\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi}}{2ir}.
\end{aligned} \tag{44}$$

[7]Euler's formula states that, for any real number ϕ, $e^{i\phi} = \cos\phi + i\sin\phi$.

Again we can substitute (38) and (39) in this expression and use polar coordinates to obtain

$$\begin{aligned}
\dot{\varphi} =& \frac{\dot{z}\, e^{-i\varphi} - \dot{w}\, e^{i\varphi}}{2ir} \\
=& \frac{(iz + a_{50}z^5 + i\, c_{32}z^3w^2 + a_{14}zw^4)\; e^{-i\varphi}}{2ir} \\
& - \frac{(-iw + \overline{a_{50}}w^5 - i\, c_{32}z^2w^3 + \overline{a_{14}}z^4w)\; e^{i\varphi}}{2ir} \\
=& \frac{(ir\, e^{i\varphi} + a_{50}r^5\, e^{5i\varphi} + i\, c_{32}r^3\, e^{3i\varphi}r^2\, e^{-2i\varphi} + a_{14}r\, e^{i\varphi}r^4\, e^{-4i\varphi})\; e^{-i\varphi}}{2ir} \\
& - \frac{(-ir\, e^{-i\varphi} + \overline{a_{50}}r^5\, e^{-5i\varphi} - i\, c_{32}r^2\, e^{2i\varphi}r^3\, e^{-3i\varphi} + \overline{a_{14}}r^4\, e^{4i\varphi}r\, e^{-i\varphi})\; e^{i\varphi}}{2ir} \\
=& \frac{ir + a_{50}r^5\, e^{4i\varphi} + i\, c_{32}r^5 + a_{14}r^5\, e^{-4i\varphi} + ir - \overline{a_{50}}r^5\, e^{-4i\varphi} + i\, c_{32}r^5 - \overline{a_{14}}r^5\, e^{4i\varphi}}{2ir} \\
=& 1 + r^4 \left[c_{32} + \frac{(a_{50}\, e^{4i\varphi} + a_{14}\, e^{-4i\varphi}) - (\overline{a_{50}}\, e^{-4i\varphi} + \overline{a_{14}}\, e^{4i\varphi})}{2i} \right] \\
=& 1 + r^4 \left[c_{32} + \mathrm{Im}\left(\mathrm{a}_{50}\, \mathrm{e}^{4\mathrm{i}\varphi} + \mathrm{a}_{14}\, \mathrm{e}^{-4\mathrm{i}\varphi} \right) \right].
\end{aligned}$$

This can be rewritten as

$$\dot{\varphi} = 1 + r^4 \left(c_{32} + b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi) - b_{14}\sin(4\varphi) + c_{14}\cos(4\varphi) \right), \quad (45)$$

by using notation $a_{kl} = b_{kl} + i\, c_{kl}$ and Euler's fomula.

Therefore, by using (43) and (45) we can rewrite system (38) in polar coordinates as

$$\begin{cases} \dot{r} = r^5 \left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi) + b_{14}\cos(4\varphi) + c_{14}\sin(4\varphi) \right), \\ \dot{\varphi} = 1 + r^4 \left(c_{32} + b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi) - b_{14}\sin(4\varphi) + c_{14}\cos(4\varphi) \right). \end{cases} \quad (46)$$

We also had condition

$$\overline{a_{14}a_{50}} = a_{14}a_{50}. \quad (47)$$

Let us write in polar coordinates $a_{14} := r_{14}\, e^{i\varphi_{14}}$ and $a_{50} := r_{50}\, e^{i\varphi_{50}}$. Using this notation, equality (47) can be written as

$$\begin{aligned}
r_{14}\, e^{-i\varphi_{14}} r_{50}\, e^{-i\varphi_{50}} &= r_{14}\, e^{i\varphi_{14}} r_{50}\, e^{i\varphi_{50}}, \\
e^{-i\varphi_{14}}\, e^{-i\varphi_{50}} &= e^{i\varphi_{14}}\, e^{i\varphi_{50}}, \\
e^{-i(\varphi_{14}+\varphi_{50})} &= e^{i(\varphi_{14}+\varphi_{50})}, \\
\cos(\varphi_{14}+\varphi_{50}) - i\sin(\varphi_{14}+\varphi_{50}) &= \cos(\varphi_{14}+\varphi_{50}) + i\sin(\varphi_{14}+\varphi_{50}), \\
-\sin(\varphi_{14}+\varphi_{50}) &= \sin(\varphi_{14}+\varphi_{50}), \\
\sin(\varphi_{14}+\varphi_{50}) &= 0.
\end{aligned}$$

The solutions for this equation are $\varphi_{14} + \varphi_{50} = 0$ and $\varphi_{14} + \varphi_{50} = \pi$. Let us study both cases separately:

– If $\varphi_{14} + \varphi_{50} = 0$ then $\varphi_{14} = -\varphi_{50}$, so

$$a_{14} = r_{14}\, e^{i\varphi_{14}} = r_{14}\, e^{-i\varphi_{50}} = \frac{r_{14}}{r_{50}} r_{50}\, e^{-i\varphi_{50}} = \frac{r_{14}}{r_{50}} \overline{a_{50}}.$$

Now using notation $a_{kl} = b_{kl} + i\, c_{kl}$, we have that $b_{14} = \frac{r_{14}}{r_{50}} b_{50}$ and $c_{14} = -\frac{r_{14}}{r_{50}} c_{50}$. Substituting this in system (46) we obtain

$$\begin{cases} \dot{r} = r^5 \left(1 + \frac{r_{14}}{r_{50}}\right) \left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right), \\ \dot{\varphi} = 1 + r^4 \left[c_{32} + \left(1 - \frac{r_{14}}{r_{50}}\right) \left(b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi)\right)\right]. \end{cases}$$

Let us define $G_1(\varphi) := \left(1 + \frac{r_{14}}{r_{50}}\right)\left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right)$ as the part of the equation of $\dot{r}$ such that only depends on the angular variable. Then we can write

$$\dot{r} = \frac{dr}{dt} = r^5\, G_1(\varphi) \Rightarrow r^{-5}\, dr = G_1(\varphi)\, dt.$$

Now integrating both sides

$$\int_{r_0}^{r} r^{-5}\, dr = \int_0^t G_1(\varphi)\, dt \Rightarrow -\frac{r^{-4}}{4} + \frac{r_0^{-4}}{4} = G_1(\varphi)\, t.$$

Let us define $\widetilde{r_0} := \frac{r_0^{-4}}{4}$ and isolate r,

$$\begin{aligned} -\frac{r^{-4}}{4} + \widetilde{r_0} &= G_1(\varphi)\, t, \\ r(t,\varphi) &= \frac{1}{\sqrt[4]{4\left(\widetilde{r_0} - G_1(\varphi)\, t\right)}}. \end{aligned} \tag{48}$$

Now we can use Maple to solve the differential equation on $\dot{\varphi}$ and find $\varphi(t,r)$, and then isolate t on this expression to find how t depends on φ. The result is

$$t(\varphi) = -\frac{1}{2} \frac{2K_1\sqrt{f_1} - r_{50}\arctan\left(\frac{f_2\tan(2\varphi) + f_3}{\sqrt{f_4}}\right)}{\sqrt{f_5}}, \tag{49}$$

where K_1 is an integration constant and f_i for $i = 1, 2, 3, 4, 5$ are expressions of sums and products such that $f_i = f_i(r, b_{14}, c_{14}, r_{14}, b_{50}, c_{50}, r_{50}, c_{32})$, i.e. they not depend on the angular variable φ.

Observe that, for any angle φ_0, $\tan(2(\varphi_0 + \pi)) = \tan(2\varphi_0 + 2\pi) = \tan(2\varphi_0) = \tan(2\varphi_0 - 2\pi) = \tan(2(\varphi_0 - \pi))$. Therefore, using expression (49) we see that $t(\varphi_0 + \pi) = t(\varphi_0 - \pi)$ for any angle φ_0, which implies that for any initial angle φ_0 the time $t(\varphi_0 + \pi)$ needed to go forward π is the same as the time $t(\varphi_0 - \pi)$ needed to go backward π. Furthermore, using the definition of G_1, we see that

$$
\begin{aligned}
G_1(\varphi_0 + \pi) &= \left(1 + \frac{r_{14}}{r_{50}}\right)(b_{50}\cos(4(\varphi_0 + \pi)) - c_{50}\sin(4(\varphi_0 + \pi))) \\
&= \left(1 + \frac{r_{14}}{r_{50}}\right)(b_{50}\cos(4\varphi_0 + 4\pi) - c_{50}\sin(4\varphi_0 + 4\pi)) \\
&= \left(1 + \frac{r_{14}}{r_{50}}\right)(b_{50}\cos(4\varphi_0) - c_{50}\sin(4\varphi_0)) \\
&= \left(1 + \frac{r_{14}}{r_{50}}\right)(b_{50}\cos(4\varphi_0 - 4\pi) - c_{50}\sin(4\varphi_0 - 4\pi)) \\
&= \left(1 + \frac{r_{14}}{r_{50}}\right)(b_{50}\cos(4(\varphi_0 - \pi)) - c_{50}\sin(4(\varphi_0 - \pi))) \\
&= G_1(\varphi_0 - \pi).
\end{aligned}
$$

Thus, we also have that $G_1(\varphi_0 + \pi) = G_1(\varphi_0 - \pi)$. Using this and the fact that $t(\varphi_0 + \pi) = t(\varphi_0 - \pi)$, according to expression (48) we obtain that

$$
\begin{aligned}
\frac{1}{\sqrt[4]{4\,(\widetilde{r_0} - G_1(\varphi + \pi)\,t(\varphi + \pi))}} &= \frac{1}{\sqrt[4]{4\,(\widetilde{r_0} - G_1(\varphi - \pi)\,t(\varphi - \pi))}}, \\
r(t(\varphi + \pi), \varphi + \pi) &= r(t(\varphi - \pi), \varphi - \pi).
\end{aligned}
$$

Finally, as the radius of the orbit when moving forward a π angle is the same as moving backward a π angle for any initial angle φ_0 we can conclude that under these conditions the origin of the system is a center.

– The case $\varphi_{14} + \varphi_{50} = \pi$ is analogous to the previous one. If $\varphi_{14} + \varphi_{50} = \pi$ then $\varphi_{14} = \pi - \varphi_{50}$, so

$$
a_{14} = r_{14}\, e^{i\varphi_{14}} = r_{14}\, e^{i(\pi - \varphi_{50})} = \frac{r_{14}}{r_{50}} r_{50}\, e^{i\pi} e^{-i\varphi_{50}} = -\frac{r_{14}}{r_{50}} \overline{a_{50}}.
$$

This implies $b_{14} = -\dfrac{r_{14}}{r_{50}} b_{50}$ and $c_{14} = \dfrac{r_{14}}{r_{50}} c_{50}$. Substituting this in system

(46) we obtain

$$\begin{cases} \dot{r} = r^5 \left(1 - \frac{r_{14}}{r_{50}}\right) \left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right), \\ \dot{\varphi} = 1 + r^4 \left[c_{32} + \left(1 + \frac{r_{14}}{r_{50}}\right) \left(b_{50}\sin(4\varphi) + c_{50}\cos(4\varphi)\right)\right]. \end{cases}$$

Now we define $G_2(\varphi) := \left(1 - \frac{r_{14}}{r_{50}}\right) \left(b_{50}\cos(4\varphi) - c_{50}\sin(4\varphi)\right)$ and, analogously to the previous case, by solving the equation in $\dot{r}$ we obtain

$$r(t, \varphi) = \frac{1}{\sqrt[4]{4\left(\widetilde{r_0} - G_2(\varphi)\, t\right)}}, \tag{50}$$

for a certain $\widetilde{r_0}$. Let us now solve the differential equation in $\dot{\varphi}$ using Maple and isolate the temporal variable to obtain

$$t(\varphi) = -\frac{1}{2} \frac{2K_2\sqrt{h_1} - r_{50}\arctan\left(\frac{h_2\tan(2\varphi) + h_3}{\sqrt{h_4}}\right)}{\sqrt{h_5}}, \tag{51}$$

where K_2 is an integration constant and h_i for $i = 1, 2, 3, 4, 5$ are expressions of sums and products such that $h_i = h_i(r, b_{14}, c_{14}, r_{14}, b_{50}, c_{50}, r_{50}, c_{32})$, i.e. they not depend on the angular variable φ.

Analogously to the previous case, we have that for any initial angle φ_0 then $G_2(\varphi_0 + \pi) = G_2(\varphi_0 - \pi)$ and $t(\varphi_0 + \pi) = t(\varphi_0 - \pi)$, so according to equation (50) we have that

$$\frac{1}{\sqrt[4]{4\left(\widetilde{r_0} - G_2(\varphi + \pi)\, t(\varphi + \pi)\right)}} = \frac{1}{\sqrt[4]{4\left(\widetilde{r_0} - G_2(\varphi - \pi)\, t(\varphi - \pi)\right)}},$$
$$r(t(\varphi + \pi), \varphi + \pi) = r(t(\varphi - \pi), \varphi - \pi).$$

Thus, again we have that for any initial angle φ_0 the radius when moving forward or backward a π radius is the same, which implies that the origin of the system is a center.

Finally, we conclude that the found conditions are actually the center conditions of the system. As the problem has been solved with $2n - 1 = 2 \cdot 5 - 1 = 9$ Lyapunov quantities, we have checked that Conjecture 5.3 holds for this fifth degree nonlinearity.

6 Conclusions and personal evaluation

Once finished this Final Master Project, the achieved conclusions and main ideas will be outlined here, as well as a brief personal evaluation on what the project has taught me academically speaking.

The main objective was to study the stability in a neighbourhood of the origin for a differential equation in the plane at a monodromic non-degenerate point, this is, a point with null lineal part and positive determinant. Two main problems arise from this analysis: distinguishing whether the origin is a center or a focus (center-focus problem) and determining the maximum number of limit cycles which will appear when perturbing the system (cyclicity problem). The key ideas we have studied during the project are enumerated below.

(a) Firstly, the notions of Poincaré map and Lyapunov quantities have been introduced as essential tools to analyze the problems. The Poincaré map indicates how the orbits of a system are after a whole loop, i.e. after a 2π angle, and it is directly related to the Lyapunov quantities L_k. If $L_k = 0$ for every $k \geq 1$, the Poincaré map is the identity map and the origin is then a center. Otherwise, if there exists any k such that $L_k \neq 0$, the Poincaré map is not the identity and the origin is a focus. Furthermore, the sign of the first non-zero Lyapunov quantity determines whether the focus is attracting or repelling. Thus, the problem reduces to finding these Lyapunov quantities.

(b) Due to the Hilbert Basis Theorem, we have seen that there always exists an $m \in \mathbb{N}$ such that all the Lyapunov quantities L_k for $k > m$ can be generated from the first m quantities. As a consequence, our problem would reduce to compute a finite number of Liapunov constants, but the inconvenience is that we do not know how many.

(c) In the center-focus problem resolution, given a center candidate there are not general methods to check that effectively the point is a center. For this reson we have introduced here some tools which can be useful in this case: first integral computation by means of Darboux integrability, Liénard systems, use of symmetries and a theorem for two monomial differential equations.

(d) For the ciclicity problem, we have seen how the Preparation Weierstrass Theorem allows to determine the maximum number of limit cycles which appear in a neighbourhood of the origin when slightly perturbing a system whose origin is a focus. Particularly, if L_k is the first non-zero Lyapunov quantity, we say that the focus has order k (definition extracted from [13]), and when it is perturbed at most k limit cycles will appear. However, if we wish to perturb a system whose origin is a center then the Preparation Weierstrass Theorem cannot be applied. We have seen in this case that if the ideal generated by the first m Lyapunov constants is radical then the problem can be automatically solved and a maximum of m limit cycles will appear. Nevertheless, if the ideal is not radical the problem becomes quite more difficult, and in many cases its resolution is unknown.

(e) A method for computing the Lyapunov quantities has been introduced: the Lyapunov method. This is quite a simple method computationally speaking because it only consisits in solving a linear system of equations, which furthermore is diagonal if it is written in complex coordinates. Then the method has been computationally implemented to create a program which computes the first M Lyapunov quantities. The chosen programming language has been PARI, mainly due to its execution speed.

(f) A second procedure to find the Lyapunov method has been studied: the interpolation technique. Given a differential equation, this method consists on choosing simpler differential equations, applying the Lyapunov method to them and then interpolating the obtained results to find the Lyapunov quantities of the original equation. This approach has two main advantages. Firstly, as the Lyapunov method could use huge matrices, it is more efficient to solve many simpler systems than a heavy one. Secondly, the fact that the problem is divided into many smaller problems allows the method to be parallelized. Thus, we could say that interpolation splits a cumbersome and heavy problem into many simple and fast problems which furthermore can be parallelized.

(g) For the considered problems and the Lyapunov quantities computation, parallelization becomes essential to achieve viable execution times. This is because the systems to solve quickly become enormous and they require a great effort on increasing computational efficiency. For example, in the case we have studied in this project we have achieved a reduction of approximately 90% in computation time thanks to parallelizing the problem.

(h) Finally, we have solved the center-focus and cyclicity problems for two polynomial families and we have used the implemented code to study a third example of homogeneous nonlinearities. Firstly, a cubic polynomial system has been analyzed. Using the center characterization techniques introduced above we were able to prove Theorem 5.1, which outlines the center conditions for these systems. We have also seen that the ideal generated by the two first Lyapunov quantities is radical, which allowed us to determine that the maximum number of limit cycles which can appear is two. Later, a quadratic polynomial family has been introduced, and its center conditions have been found through Kapteyn-Bautin Theorem 5.2. In this case, when studying the ciclicity problem we see that the ideal generated by the three first Lyapunov quantities is not radical, so the problem cannot be solved automatically as we have seen. This problem was solved by Bautin in [3], where it is proved that the maximum number of limit cycles which can appear in a neighbourhood of the origin when perturbing the system is three. The third example we have seen is about homogeneous nonlinearities. A conjecture has been outlined, which states that for degree n homogeneous polynomial nonlinearities the minimum number of Lyapunov quantities needed to solve the center-focus problem is $2n - 1$. This statement has been checked for a fifth degree family by using the implemented code to compute the corresponding Lyapunov quantities and finding the center conditions to see that the needed number of Lyapunov quantities is $2n - 1 = 2 \cdot 5 - 1 = 9$, as the conjecture predicts.

The fact that there are not standard methods to solve the center-focus and ciclicity problems makes that the resolution of these problems is known only for a few polynomial families. In this sense, the problem is open and there is still a lot of work to be done on it and on the computation of the Lyapunov quantities. For this reason, this Final Master Project is only a first step of what I expect to do during my PhD in the following years, in which I intend for example to complete the parallelization of the interpolation technique or to carry out deeper studies on limit cycles.

As a personal evaluation I can state that I have achieved the objectives introduced at the beginning of the project. First, I consider that I have improved my research capacity by learning to search in articles and bibliography the resources I needed, as well as to express information with the proper mathematical formalism. This research has also allowed me to consolidate and increase my knowledge on dynamical systems and other branches of mathematics and computation. Furthermore, I have checked how different fields of mathematics work together with computer science to achieve their goals more efficiently. With the elaborated codes I have shown how specific mathematical procedures which might be arduous to execute can be automatized by using computer programming.

In general lines, the temporary planning proposed in Figure 2 has been successfully followed. Even though the exact dates have not been fulfilled as I planned due to unexpected problems or drawbacks, the structure of the planning and the methodology have turned out to be suitable for the development of the project, and this has allowed me to deliver the project fully finished and reviewed on time.

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Appendices

Appendix A

Implemented PARI code

The PARI code implemented in file `lyapunov_quantities.gp` is as follows:

```
allocatemem(4096000000);

complex(X,Y)=
{
  -I*z+substvec(X+I*Y,[x,y],[(z+w)/2,(z-w)/(2*I)]);
}

conjugate(Z)=
{
  conj(substvec(Z,[z,w],[w,z]));
}

complex_product(a,b)=
{
  (real(a)*real(b)-imag(a)*imag(b))+
    I*(real(a)*imag(b)+imag(a)*real(b));
}

lyapunov_solve(Z,M)=
{
  local(p,phi);
  W=conjugate(Z);
  L=vector(M);
  h=matrix(2*M+3,2*M+3);
  h[2,2]=1/2;
  p=2;
  for(n=1,M,
    p++;
```

```
    phi=sum(k=2,p-1,complex_product(deriv(sum(i=0,p-k+1,
      h[p-k+1-i+1,i+1]*z^(p-k+1-i)*w^i),z),sum(i=0,k,polcoeff
      (polcoeff(Z,i,w),k-i,z)*z^(k-i)*w^i))+complex_product(deriv
      (sum(i=0,p-k+1,h[p-k+1-i+1,i+1]*z^(p-k+1-i)*w^i),w),
      conjugate(sum(i=0,k,polcoeff(polcoeff(Z,i,w),k-i,z)
      *z^(k-i)*w^i))));
    for(j=0,p,h[p-j+1,j+1]=-I*polcoeff(polcoeff(phi,p-j,z),j,w)
      /(2*j-p));
    p++;
    phi=sum(k=2,p-1,complex_product(deriv(sum(i=0,p-k+1,
      h[p-k+1-i+1,i+1]*z^(p-k+1-i)*w^i),z),sum(i=0,k,polcoeff(
      polcoeff(Z,i,w),k-i,z)*z^(k-i)*w^i))+complex_product(deriv
      (sum(i=0,p-k+1,h[p-k+1-i+1,i+1]*z^(p-k+1-i)*w^i),w),
      conjugate(sum(i=0,k,polcoeff(polcoeff(Z,i,w),k-i,z)
      *z^(k-i)*w^i))));
    for(j=0,p,if(j!=p/2,h[p-j+1,j+1]=-I*polcoeff(polcoeff(phi,p-j,z),
      j,w)/(2*j-p)));
    L[p/2-1]=polcoeff(polcoeff(phi,p/2,z),p/2,w);
  );
  L;
}


deg(r)=
{
  sum(i=1,matsize(r)[1],r[i,4]);
}

qdeg(r)=
{
  sum(i=1,matsize(r)[1],(r[i,1]+r[i,2]-1)*r[i,4]);
}

weight(r)=
{
  sum(i=1,matsize(r)[1],(1-r[i,1]+r[i,2])*r[i,3]*r[i,4]);
}

check_element(a,b)=
/* This routine checks:
   - if two matrices represent the same monomial, i.e. they only
     differ in the order of their rows
```

```
   - if two matrices represent conjugate monomials,  i.e. the
     signs of the values in the third column are all opposite. */
{
  EQ=1;
  /* If EQ=1 then a and b are equal or conjugate;
       if EQ=0 they are not */
  if(matsize(a)[1]!=matsize(b)[1],EQ=0,
    for(i=1,matsize(a)[1],
      for(j=1,matsize(b)[1],
        if(a[i,]==b[j,],break(n=1));
        if(j==matsize(b)[1],
        /* Row a[i,] is not found in b, so a and b are not equal */
          EQ=0;
          break(n=2);
        );
      );
    );
    if(EQ==0,
      NOTCONJ=0;
      for(i=1,matsize(a)[1],
        for(j=1,matsize(b)[1],
          if(a[i,1]==b[j,1] && a[i,2]==b[j,2] && a[i,3]==-b[j,3]
            && a[i,4]==b[j,4],break(n=1));
          if(j==matsize(b)[1],
          /* Conjugate of row a[i,] is not found in b,
              so a and b are not conjugate */
            NOTCONJ=1;
            break(n=2);
          );
        );
        ITER=i;
      );
      if(ITER==matsize(a)[1] && NOTCONJ==0,EQ=1);
    );
  );
  EQ;
}

check_list(r,L)=
/* This routine checks if monomial r or its conjugat
    is in the list L */
{
```

```
  ret=0;
  for(k=1,length(L),
    ret=check_element(r,L[k]);
    if(ret==1,break(n=1));
  );
  ret;
}

arrange_monomial(r)=
/* This function groups factors which are equal and assigns
    them their corresponding degree */
{
  FINISH=0;
  while(FINISH==0,
    FINISH=1;
    DIM=matsize(r)[1];
    for(i=1,DIM,
      for(j=1,DIM,
        if(r[i,1..3]==r[j,1..3] && i!=j,
          r[i,4]=r[i,4]+r[j,4];
          if((j+1)<=DIM,r=concat(r[1..(j-1),]~,r[(j+1)..DIM,]~)~,
            r=r[1..(j-1),]);
          FINISH=0;
          break(n=2);
        );
      );
    );
  );
  r;
}

theoremA(monomials)=
{
  monomialsA=listcreate(0);
  for(i=1,length(monomials),
    r=monomials[i];
    INSERT=1;
    if((matsize(r)[1]%2)==0,
      R=r;
      for(j=1,matsize(r)[1]/2,
        for(k=j+1,matsize(R)[1],
          if(R[j,1]==R[k,1] && R[j,2]==R[k,2] && R[j,4]==R[k,4],
```

```
            if((k+1)<=matsize(R)[1],R=concat(R[1..(k-1),]~,
              R[(k+1)..matsize(R)[1],]~)~,R=R[1..(k-1),]);
            INSERT=0;
            break(n=1);
          );
        );
      );
    );
    if(INSERT==1 || matsize(R)[1]!=matsize(monomials[i])[1]/2,
      listinsert(monomialsA,monomials[i],length(monomialsA)+1));
  );
  monomialsA;
}

create_monomials(par,M)=
/* Creates the monomials list for the M-th Lyapunov quantity
    using the parameters in par; par must be a list of row
    vectors [i,j] which contains the subindices */
{
  r=listcreate(0);
  for(k=1,length(par),
    listinsert(r,Mat([par[k][1],par[k][2],-1,1]),1);
    listinsert(r,Mat([par[k][1],par[k][2],1,1]),1);
  );
  m=length(r);
  pos=listcreate(0);
  monomials=listcreate(0);
  forpart(d=2*M,
    L=listcreate(0);
    max_pos=listcreate(0);
    N=0;
    for(i=1,length(d),
      cont=0;
      for(j=1,m,
        if(qdeg(r[j])==d[i],
          listinsert(L,r[j],length(L)+1);
          cont=cont+1;
        );
      );
      if(cont!=0,
        N=N+1;
        listinsert(max_pos,cont,length(max_pos)+1)
```

```
      );
    );
    if(length(L)>0 && length(max_pos)==length(d),
      for(l=1,N,listinsert(pos,1,1));
      while(true,
        j=N;
        new_monomial=L[pos[1]]~;
        for(t=2,N,new_monomial=concat(new_monomial,
          L[sum(v=1,t-1,max_pos[v])+pos[t]]~));
        new_monomial=arrange_monomial(new_monomial~);
        if(check_list(new_monomial,monomials)==0 &&
          weight(new_monomial)==0 && qdeg(new_monomial)==2*M,
          listinsert(monomials,new_monomial,length(monomials)+1));
        if(pos[j]!=max_pos[j],pos[j]=pos[j]+1,
          while(pos[j]==max_pos[j],
            if(j==1,
              break(n=2);
            );
            if(j!=1,
              if(pos[j-1]==max_pos[j-1],j=j-1,
                pos[j-1]=pos[j-1]+1;
                for(s=j,N,pos[s]=1);
                break(n=1);
              );
            );
          );
        );
      );
    );
    listkill(pos);
  );
  theoremA(monomials);
}

convert_monomial(r)=
/* Converts the mononomial matrix into monomial */
{
  M=1;
  for(i=1,matsize(r)[1],
    if(r[i,3]==1,M=M*eval(Str("r",r[i,1],r[i,2]))^r[i,4],
      M=M*eval(Str("Cr",r[i,1],r[i,2]))^r[i,4]);
  );
```

```
  M;
}

conjugate_monomial(r)=
{
  for(i=1,matsize(r)[1],r[i,3]=-r[i,3]);
  r;
}

generate_equation(r)=
{
  Z=I*z+sum(j=1,matsize(r)[1],((random(21)-10)+
  (random(21)-10)*I)*z^r[j,1]*w^r[j,2]);
}


interpolation(par,M,syst)=
/* Interpolation method to compute the M-th Lyapunov constant
    of a system with parameters par; par is a list of subindices;
    syst is the list/vector of differential equations to perform the
    interpolation; if syst==0 the equations are randomly generated,
    and if there are not enough equations the rest of equations also
    are randomly generated (this may lead to sparse systems) */
{
  mon=create_monomials(par,M);
  for(i=1,length(mon),listinsert(mon,conjugate_monomial(mon[i]),
    length(mon)+1));
  terms=listcreate(0);
  for(i=1,length(mon)/2,
    listinsert(terms,(eval(Str("a",i))+I*eval(Str("b",i)))*
      convert_monomial(mon[i]),length(terms)+1);
    listinsert(terms,(eval(Str("a",i))-I*eval(Str("b",i)))*
      convert_monomial(mon[length(mon)/2+i]),length(terms)+1);
  );
  A=matrix(length(terms),length(terms));
  y=vector(length(terms))~;
  count=0;
  while(matrank(A)!=length(terms),
    count++;
    if(count==1000,break(n=1));
    for(i=1,length(terms),
      S=sum(j=1,length(terms),terms[j]);
```

```
      if(syst==0 || i>length(syst),Z=generate_equation(mon[i]),
        Z=syst[i]);
      y[i]=lyapunov_method(Z,M)[M];
      for(k=1,length(par),
        c=polcoeff(polcoeff(Z,par[k][1],z),par[k][2],w);
        S=subst(S,eval(Str("r",par[k][1],par[k][2])),c);
        S=subst(S,eval(Str("Cr",par[k][1],par[k][2])),conj(c));
      );
      for(j=1,length(terms)/2,
        A[i,j]=polcoeff(S,1,eval(Str("a",j)));
        A[i,length(terms)/2+j]=polcoeff(S,1,eval(Str("b",j)));
      );
    );
    if(matrank(A)!=length(terms) && syst!=0,
      print("The introduced equations lead to an underdetermined
        system...");
      count=1000;
      break(n==1);
    );
  );
  if(count==1000,print("A suitable system cannot be generated..."),
    u=matsolve(A,y);
    L=sum(i=1,length(terms),terms[i]);
    for(i=1,length(u)/2,
      L=subst(L,eval(Str("a",i)),u[i]);
      L=subst(L,eval(Str("b",i)),u[length(u)/2+i]);
    );
    L;
  );
}
```

Appendix B

Data file for parallelization

Data file with the 28 polynomials for the Lyapunov method parallelization:

```
1,z^3*w^2
2,z^3-I*z*w^2
3,z^2-I*z^3*w
4,z^2+I*z^2*w^2
5,z*w-I*z^3*w
6,z*w+I*z^2*w^2
7,w^2+I*z^4
8,w^2+I*z*w^3
9,z^2+z*w^2
10,z*w-z^3
11,z*w+z*w^2
12,z^2+z*w+z*w^2
13,z^2+z*w-z^3
14,z^2+w^2-z^3
15,z^2+w^2+z*w^2
16,z*w+w^2+z*w^2
17,z*w+w^2+z^3
18,z^2+w^2+w^3
19,z*w+w^2+w^3
20,I*z^2+w^2
21,z*w+I*w^2
22,z^2+I*z*w+(1+I)*z^2*w
23,z^2+(1+I)*z*w+(1+I)*z^2*w
24,(1+I)*z^2+z*w+(1+I)*z^2*w
25,z^2+(1+I)*z*w+z^2*w
26,z^2-I*z*w+w^2-z^2*w
27,I*z^2+z*w+w^2+z^2*w
28,I*z^2+I*z*w+w^2
```

Appendix C

Lyapunov quantities for the fifth degree homogeneous nonlinearities

First 20 Lyapunov quantities for equation $\dot{z} = iz + a_{50}z^5 + a_{32}z^3w^2 + a_{14}zw^4$.

$$L_1 = 0,$$

$$L_2 = 1/2\, a_{32} + 1/2\, \overline{a_{32}},$$

$$L_3 = 0,$$

$$L_4 = (1/2\, i\, a_{14}\, a_{50} - 1/2\, i\, \overline{a_{14}}\, \overline{a_{50}}),$$

$$L_5 = 0,$$

$$L_6 = ((-9/16\, \overline{a_{50}} - 19/16\, a_{14})\, a_{50} + (-11/16\, \overline{a_{14}}\, \overline{a_{50}} - 21/16\, \overline{a_{14}}\, a_{14}))\, a_{32} + ((-9/16\, \overline{a_{50}} - 11/16\, a_{14})\, a_{50} + (-19/16\, \overline{a_{14}}\, \overline{a_{50}} - 21/16\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}},$$

$$L_7 = 0,$$

$$L_8 = ((-5/4\, i\, \overline{a_{50}} - 75/32\, i\, a_{14})\, a_{50} + (-37/32\, i\, \overline{a_{14}}\, \overline{a_{50}} - 9/4\, i\, \overline{a_{14}}\, a_{14}))\, a_{32}^2 + (-11/16\, i\, a_{14}\, a_{50} + 11/16\, i\, \overline{a_{14}}\, \overline{a_{50}})\, \overline{a_{32}}\, a_{32} + (((5/4\, i\, \overline{a_{50}} + 37/32\, i\, a_{14})\, a_{50} + (75/32\, i\, \overline{a_{14}}\, \overline{a_{50}} + 9/4\, i\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^2 + ((-15/16\, i\, a_{14}\, \overline{a_{50}} - 25/16\, i\, a_{14}^2)\, a_{50}^2 + (15/16\, i\, \overline{a_{14}}\, \overline{a_{50}}^2 - 27/16\, i\, \overline{a_{14}}\, a_{14}^2)\, a_{50} + (25/16\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^2 + 27/16\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}}))),$$

$$L_9 = 0,$$

$$L_{10} = ((1155/256\, \overline{a_{50}} + 1785/256\, a_{14})\, a_{50} + (1185/256\, \overline{a_{14}}\, \overline{a_{50}} + 1815/256\, \overline{a_{14}}\, a_{14}))\, a_{32}^3 + ((1785/256\, \overline{a_{50}} + 2531/256\, a_{14})\, a_{50} + (2059/256\, \overline{a_{14}}\, \overline{a_{50}} + 2805/256\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}\, a_{32}^2 + (((1785/256\, \overline{a_{50}} + 2059/256\, a_{14})\, a_{50} + (2531/256\, \overline{a_{14}}\, \overline{a_{50}} + 2805/256\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^2 + ((1897/1024\, \overline{a_{50}}^2 + 5069/512\, a_{14}\, \overline{a_{50}} + 11521/1024\, a_{14}^2)\, a_{50}^2 + (2445/512\, \overline{a_{14}}\, \overline{a_{50}}^2$$

$$+4029/256\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} + 7709/512\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (4097/1024\,\overline{a_{14}}^2$$
$$\overline{a_{50}}^2 + 3677/512\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} + 4169/1024\,\overline{a_{14}}^2\,a_{14}^2)))\,a_{32}+$$
$$(((1155/256\,\overline{a_{50}} + 1185/256\,a_{14})\,a_{50} + (1785/256\,\overline{a_{14}}\,\overline{a_{50}} + 1815/256\,\overline{a_{14}}$$
$$a_{14}))\,\overline{a_{32}}^3 + ((1897/1024\,\overline{a_{50}}^2 + 2445/512\,a_{14}\,\overline{a_{50}} + 4097/1024\,a_{14}^2)$$
$$a_{50}^2 + (5069/512\,\overline{a_{14}}\,\overline{a_{50}}^2 + 4029/256\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} + 3677/512$$
$$\overline{a_{14}}\,a_{14}^2)\,a_{50} + (11521/1024\,\overline{a_{14}}^2\,\overline{a_{50}}^2 + 7709/512\,\overline{a_{14}}^2\,a_{14}$$
$$\overline{a_{50}} + 4169/1024\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}),$$
$$L_{11} = 0,$$
$$L_{12} = ((945/64\,i\,\overline{a_{50}} + 11025/512\,i\,a_{14})\,a_{50} + (7455/512\,i\,\overline{a_{14}}\,\overline{a_{50}} + 1365/64$$
$$i\,\overline{a_{14}}\,a_{14}))\,a_{32}^4 + ((729/32\,i\,\overline{a_{50}} + 4493/128\,i\,a_{14})\,a_{50} + (2635/128\,i\,\overline{a_{14}}$$
$$\overline{a_{50}} + 1053/32\,i\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}\,a_{32}^3 + ((2059/256\,i\,a_{14}\,a_{50} - 2059/256\,i$$
$$\overline{a_{14}}\,\overline{a_{50}})\,\overline{a_{32}}^2 + ((5589/512\,i\,\overline{a_{50}}^2 + 60397/1024\,i\,a_{14}\,\overline{a_{50}}+$$
$$63289/1024\,i\,a_{14}^2)\,a_{50}^2 + (9047/1024\,i\,\overline{a_{14}}\,\overline{a_{50}}^2 + 4617/64\,i\,\overline{a_{14}}\,a_{14}$$
$$\overline{a_{50}} + 83661/1024\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-4393/1024\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2+$$
$$10575/1024\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} + 9867/512\,i\,\overline{a_{14}}^2\,a_{14}^2)))\,a_{32}^2 + (((-729/32\,i$$
$$\overline{a_{50}} - 2635/128\,i\,a_{14})\,a_{50} + (-4493/128\,i\,\overline{a_{14}}\,\overline{a_{50}} - 1053/32\,i\,\overline{a_{14}}$$
$$a_{14}))\,\overline{a_{32}}^3 + ((17451/512\,i\,a_{14}\,\overline{a_{50}} + 23185/512\,i\,a_{14}^2)\,a_{50}^2 + (-17451/512\,i$$
$$\overline{a_{14}}\,\overline{a_{50}}^2 + 24991/512\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-23185/512\,i\,\overline{a_{14}}^2$$
$$\overline{a_{50}}^2 - 24991/512\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}))\,\overline{a_{32}})\,a_{32} + (((-945/64\,i\,\overline{a_{50}}$$
$$-7455/512\,i\,a_{14})\,a_{50} + (-11025/512\,i\,\overline{a_{14}}\,\overline{a_{50}} - 1365/64\,i\,\overline{a_{14}}\,a_{14}))$$
$$\overline{a_{32}}^4 + ((-5589/512\,i\,\overline{a_{50}}^2 - 9047/1024\,i\,a_{14}\,\overline{a_{50}} + 4393/1024\,i\,a_{14}^2)\,a_{50}^2+$$
$$(-60397/1024\,i\,\overline{a_{14}}\,\overline{a_{50}}^2 - 4617/64\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} - 10575/1024\,i$$
$$\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-63289/1024\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2 - 83661/1024\,i\,\overline{a_{14}}^2\,a_{14}$$
$$\overline{a_{50}} - 9867/512\,i\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^2 + ((4185/1024\,i\,a_{14}\,\overline{a_{50}}^2$$
$$+7335/512\,i\,a_{14}^2\,\overline{a_{50}} + 12645/1024\,i\,a_{14}^3)\,a_{50}^3 + (-4185/1024\,i\,\overline{a_{14}}\,\overline{a_{50}}^3$$
$$+20735/1024\,i\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} + 10011/512\,i\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 + (-7335/512$$
$$i\,\overline{a_{14}}^2\,\overline{a_{50}}^3 - 20735/1024\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2 + 7553/1024\,i$$
$$\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (-12645/1024\,i\,\overline{a_{14}}^3\,\overline{a_{50}}^3 - 10011/512\,i\,\overline{a_{14}}^3$$
$$a_{14}\,\overline{a_{50}}^2 - 7553/1024\,i\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}))),$$
$$L_{13} = 0,$$
$$L_{14} = ((-252945/4096\,\overline{a_{50}} - 343035/4096\,a_{14})\,a_{50} + (-254835/4096\,\overline{a_{14}}\,\overline{a_{50}}-$$
$$344925/4096\,\overline{a_{14}}\,a_{14}))\,a_{32}^5 + ((-685685/4096\,\overline{a_{50}} - 912063/4096\,a_{14})\,a_{50}+$$
$$(-708647/4096\,\overline{a_{14}}\,\overline{a_{50}} - 935025/4096\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}\,a_{32}^4 + (((-445445/2048$$
$$\overline{a_{50}} - 555751/2048\,a_{14})\,a_{50} + (-497119/2048\,\overline{a_{14}}\,\overline{a_{50}} - 607425/2048\,\overline{a_{14}}$$

$$a_{14}))\,\overline{a_{32}}^2 + ((-5817427/65536\,\overline{a_{50}}^2 - 12083539/32768\,a_{14}\,\overline{a_{50}} - 22244931/65536$$
$$a_{14}^2)\,a_{50}^2 + (-4854963/32768\,\overline{a_{14}}\,\overline{a_{50}}^2 - 9017575/16384\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}$$
$$-15882027/32768\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-4252547/65536\,\overline{a_{14}}^2\,\overline{a_{50}}^2 - 6187531/32768$$
$$\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} - 9630915/65536\,\overline{a_{14}}^2\,a_{14}^2)))\,a_{32}^3 + (((-445445/2048\,\overline{a_{50}}-$$
$$497119/2048\,a_{14})\,a_{50} + (-555751/2048\,\overline{a_{14}}\,\overline{a_{50}} - 607425/2048\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^3$$
$$+((-9430905/65536\,\overline{a_{50}}^2 - 17369177/32768\,a_{14}\,\overline{a_{50}} - 30726025/65536\,a_{14}^2)\,a_{50}^2+$$
$$(-11400121/32768\,\overline{a_{14}}\,\overline{a_{50}}^2 - 15395381/16384\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} - 23096929/32768$$
$$\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-15826121/65536\,\overline{a_{14}}^2\,\overline{a_{50}}^2 - 15071553/32768\,\overline{a_{14}}^2\,a_{14}$$
$$\overline{a_{50}} - 16310985/65536\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}})\,a_{32}^2 + (((-685685/4096\,\overline{a_{50}}-$$
$$708647/4096\,a_{14})\,a_{50} + (-912063/4096\,\overline{a_{14}}\,\overline{a_{50}} - 935025/4096\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^4$$
$$+((-9430905/65536\,\overline{a_{50}}^2 - 11400121/32768\,a_{14}\,\overline{a_{50}} - 15826121/65536\,a_{14}^2)\,a_{50}^2$$
$$+(-17369177/32768\,\overline{a_{14}}\,\overline{a_{50}}^2 - 15395381/16384\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} - 15071553/32768$$
$$\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-30726025/65536\,\overline{a_{14}}^2\,\overline{a_{50}}^2 - 23096929/32768$$
$$\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} - 16310985/65536\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^2 + ((-104159/8192$$
$$\overline{a_{50}}^3 - 2430917/24576\,a_{14}\,\overline{a_{50}}^2 - 2000793/8192\,a_{14}^2\,\overline{a_{50}} - 4432579/24576\,a_{14}^3)$$
$$a_{50}^3 + (-1247789/24576\,\overline{a_{14}}\,\overline{a_{50}}^3 - 5469169/24576\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2-$$
$$11049379/24576\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} - 2590357/8192\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 + (-831017/8192$$
$$\overline{a_{14}}^2\,\overline{a_{50}}^3 - 1983273/8192\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2 - 2351599/8192\,\overline{a_{14}}^2$$
$$a_{14}^2\,\overline{a_{50}} - 1352463/8192\,\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (-1791115/24576\,\overline{a_{14}}^3\,\overline{a_{50}}^3$$
$$-3306671/24576\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2 - 2194133/24576\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}-$$
$$247835/8192\,\overline{a_{14}}^3\,a_{14}^3)))\,a_{32} + (((-252945/4096\,\overline{a_{50}} - 254835/4096\,a_{14})\,a_{50}+$$
$$(-343035/4096\,\overline{a_{14}}\,\overline{a_{50}} - 344925/4096\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^5 + ((-5817427/65536\,\overline{a_{50}}^2$$
$$-4854963/32768\,a_{14}\,\overline{a_{50}} - 4252547/65536\,a_{14}^2)\,a_{50}^2 + (-12083539/32768\,\overline{a_{14}}\,\overline{a_{50}}^2$$
$$-9017575/16384\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} - 6187531/32768\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-22244931/65536$$
$$\overline{a_{14}}^2\,\overline{a_{50}}^2 - 15882027/32768\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} - 9630915/65536\,\overline{a_{14}}^2$$
$$a_{14}^2))\,\overline{a_{32}}^3 + ((-104159/8192\,\overline{a_{50}}^3 - 1247789/24576\,a_{14}\,\overline{a_{50}}^2 - 831017/8192\,a_{14}^2$$
$$\overline{a_{50}} - 1791115/24576\,a_{14}^3)\,a_{50}^3 + (-2430917/24576\,\overline{a_{14}}\,\overline{a_{50}}^3 - 5469169/24576$$
$$\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2 - 1983273/8192\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} - 3306671/24576\,\overline{a_{14}}$$
$$a_{14}^3)\,a_{50}^2 + (-2000793/8192\,\overline{a_{14}}^2\,\overline{a_{50}}^3 - 11049379/24576\,\overline{a_{14}}^2\,a_{14}$$
$$\overline{a_{50}}^2 - 2351599/8192\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} - 2194133/24576\,\overline{a_{14}}^2\,a_{14}^3)\,a_{50}$$
$$+(-4432579/24576\,\overline{a_{14}}^3\,\overline{a_{50}}^3 - 2590357/8192\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2 - 1352463/8192$$
$$\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}} - 247835/8192\,\overline{a_{14}}^3\,a_{14}^3))\,\overline{a_{32}}),$$
$$L_{15} = 0,$$

$$
\begin{gathered}
L_{16} = ((-135135/512\, i\, \overline{a_{50}} - 2837835/8192\, i\, a_{14})\, a_{50} + (-2151765/8192\, i\, \overline{a_{14}}\, \overline{a_{50}} - \\
176715/512\, i\, \overline{a_{14}}\, a_{14}))\, a_{32}^6 + ((-104247/128\, i\, \overline{a_{50}} - 4432569/4096\, i\, a_{14})\, a_{50} + \\
(-3265671/4096\, i\, \overline{a_{14}}\, \overline{a_{50}} - 136323/128\, i\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}\, a_{32}^5 + (((-438555/512\, i \\
\overline{a_{50}} - 9916549/8192\, i\, a_{14})\, a_{50} + (-6276251/8192\, i\, \overline{a_{14}}\, \overline{a_{50}} - 573495/512\, i\, \overline{a_{14}} \\
a_{14}))\, \overline{a_{32}}^2 + ((-17511273/32768\, i\, \overline{a_{50}}^2 - 2186931/1024\, i\, a_{14}\, \overline{a_{50}} - 7675383/4096\, i\, a_{14}^2) \\
a_{50}^2 + (-12012879/16384\, i\, \overline{a_{14}}\, \overline{a_{50}}^2 - 3118575/1024\, i\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}} \\
-691509/256\, i\, \overline{a_{14}}\, a_{14}^2)\, a_{50} + (-1495641/8192\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^2 - 14551569/16384\, i \\
\overline{a_{14}}^2\, a_{14}\, \overline{a_{50}} - 26932539/32768\, i\, \overline{a_{14}}^2\, a_{14}^2)))\, a_{32}^4 + ((-497119/2048\, i\, a_{14} \\
a_{50} + 497119/2048\, i\, \overline{a_{14}}\, \overline{a_{50}})\, \overline{a_{32}}^3 + ((-13650377/16384\, i\, \overline{a_{50}}^2 - 63754641/16384 \\
i\, a_{14}\, \overline{a_{50}} - 29601113/8192\, i\, a_{14}^2)\, a_{50}^2 + (-10527149/16384\, i\, \overline{a_{14}}\, \overline{a_{50}}^2 - 2469759/512 \\
i\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}} - 81337599/16384\, i\, \overline{a_{14}}\, a_{14}^2)\, a_{50} + (2678027/8192\, i\, \overline{a_{14}}^2 \\
\overline{a_{50}}^2 - 12228451/16384\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}} - 21319003/16384\, i\, \overline{a_{14}}^2\, a_{14}^2))\, \overline{a_{32}}) \\
a_{32}^3 + (((438555/512\, i\, \overline{a_{50}} + 6276251/8192\, i\, a_{14})\, a_{50} + (9916549/8192\, i\, \overline{a_{14}}\, \overline{a_{50}} + \\
573495/512\, i\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^4 + ((-31529475/16384\, i\, a_{14}\, \overline{a_{50}} - 19181775/8192\, i \\
a_{14}^2)\, a_{50}^2 + (31529475/16384\, i\, \overline{a_{14}}\, \overline{a_{50}}^2 - 41032301/16384\, i\, \overline{a_{14}}\, a_{14}^2)\, a_{50} \\
+(19181775/8192\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^2 + 41032301/16384\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}}))\, \overline{a_{32}}^2 \\
+((-267111/2048\, i\, \overline{a_{50}}^3 - 223680493/196608\, i\, a_{14}\, \overline{a_{50}}^2 - 86421813/32768\, i\, a_{14}^2\, \overline{a_{50}} - \\
353804321/196608\, i\, a_{14}^3)\, a_{50}^3 + (-18311059/196608\, i\, \overline{a_{14}}\, \overline{a_{50}}^3 - 34064753/18432\, i\, \overline{a_{14}} \\
a_{14}\, \overline{a_{50}}^2 - 310308369/65536\, i\, \overline{a_{14}}\, a_{14}^2\, \overline{a_{50}} - 969295441/294912\, i\, \overline{a_{14}} \\
a_{14}^3)\, a_{50}^2 + (10666133/32768\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^3 - 23251055/65536\, i\, \overline{a_{14}}^2\, a_{14} \\
\overline{a_{50}}^2 - 4626399/2048\, i\, \overline{a_{14}}^2\, a_{14}^2\, \overline{a_{50}} - 114165751/65536\, i\, \overline{a_{14}}^2\, a_{14}^3) \\
a_{50} + (64154017/196608\, i\, \overline{a_{14}}^3\, \overline{a_{50}}^3 + 124628849/294912\, i\, \overline{a_{14}}^3\, a_{14}\, \overline{a_{50}}^2 \\
-8374153/65536\, i\, \overline{a_{14}}^3\, a_{14}^2\, \overline{a_{50}} - 4608785/18432\, i\, \overline{a_{14}}^3\, a_{14}^3)))\, a_{32}^2 + \\
(((104247/128\, i\, \overline{a_{50}} + 3265671/4096\, i\, a_{14})\, a_{50} + (4432569/4096\, i\, \overline{a_{14}}\, \overline{a_{50}} + 136323/128 \\
i\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^5 + ((13650377/16384\, i\, \overline{a_{50}}^2 + 10527149/16384\, i\, a_{14}\, \overline{a_{50}} - \\
2678027/8192\, i\, a_{14}^2)\, a_{50}^2 + (63754641/16384\, i\, \overline{a_{14}}\, \overline{a_{50}}^2 + 2469759/512\, i\, \overline{a_{14}} \\
a_{14}\, \overline{a_{50}} + 12228451/16384\, i\, \overline{a_{14}}\, a_{14}^2)\, a_{50} + (29601113/8192\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^2 + \\
81337599/16384\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}} + 21319003/16384\, i\, \overline{a_{14}}^2\, a_{14}^2))\, \overline{a_{32}}^3 + \\
((-24889647/32768\, i\, a_{14}\, \overline{a_{50}}^2 - 35965125/16384\, i\, a_{14}^2\, \overline{a_{50}} - 52332779/32768\, i\, a_{14}^3)\, a_{50}^3 \\
+(24889647/32768\, i\, \overline{a_{14}}\, \overline{a_{50}}^3 - 319105043/98304\, i\, \overline{a_{14}}\, a_{14}^2\, \overline{a_{50}} - 45470329/16384\, i \\
\overline{a_{14}}\, a_{14}^3)\, a_{50}^2 + (35965125/16384\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^3 + 319105043/98304\, i\, \overline{a_{14}}^2 \\
a_{14}\, \overline{a_{50}}^2 - 118066949/98304\, i\, \overline{a_{14}}^2\, a_{14}^3)\, a_{50} + (52332779/32768\, i\, \overline{a_{14}}^3\, \overline{a_{50}}^3 \\
+45470329/16384\, i\, \overline{a_{14}}^3\, a_{14}\, \overline{a_{50}}^2 + 118066949/98304\, i\, \overline{a_{14}}^3\, a_{14}^2\, \overline{a_{50}})) \\
\overline{a_{32}})\, a_{32} + (((135135/512\, i\, \overline{a_{50}} + 2151765/8192\, i\, a_{14})\, a_{50} + (2837835/8192\, i\, \overline{a_{14}}
\end{gathered}
$$

$\overline{a_{50}} + 176715/512\, i\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^6 + ((17511273/32768\, i\, \overline{a_{50}}^2 + 12012879/16384$
$i\, a_{14}\, \overline{a_{50}} + 1495641/8192\, i\, a_{14}^2)\, a_{50}^2 + (2186931/1024\, i\, \overline{a_{14}}\, \overline{a_{50}}^2 + 3118575/1024$
$i\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}} + 14551569/16384\, i\, \overline{a_{14}}\, a_{14}^2)\, a_{50} + (7675383/4096\, i\, \overline{a_{14}}^2$
$\overline{a_{50}}^2 + 691509/256\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}} + 26932539/32768\, i\, \overline{a_{14}}^2\, a_{14}^2))$
$\overline{a_{32}}^4 + ((267111/2048\, i\, \overline{a_{50}}^3 + 18311059/196608\, i\, a_{14}\, \overline{a_{50}}^2 - 10666133/32768\, i\, a_{14}^2$
$\overline{a_{50}} - 64154017/196608\, i\, a_{14}^3)\, a_{50}^3 + (223680493/196608\, i\, \overline{a_{14}}\, \overline{a_{50}}^3 + 34064753/18432\, i$
$\overline{a_{14}}\, a_{14}\, \overline{a_{50}}^2 + 23251055/65536\, i\, \overline{a_{14}}\, a_{14}^2\, \overline{a_{50}} - 124628849/294912\, i\, \overline{a_{14}}$
$a_{14}^3)\, a_{50}^2 + (86421813/32768\, i\, \overline{a_{14}}^2\, \overline{a_{50}}^3 + 310308369/65536\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}}^2$
$+4626399/2048\, i\, \overline{a_{14}}^2\, a_{14}^2\, \overline{a_{50}} + 8374153/65536\, i\, \overline{a_{14}}^2\, a_{14}^3)\, a_{50} + (353804321/196608$
$i\, \overline{a_{14}}^3\, \overline{a_{50}}^3 + 969295441/294912\, i\, \overline{a_{14}}^3\, a_{14}\, \overline{a_{50}}^2 + 114165751/65536\, i$
$\overline{a_{14}}^3\, a_{14}^2\, \overline{a_{50}} + 4608785/18432\, i\, \overline{a_{14}}^3\, a_{14}^3))\, \overline{a_{32}}^2 + ((-274365/8192\, i$
$a_{14}\, \overline{a_{50}}^3 - 1398345/8192\, i\, a_{14}^2\, \overline{a_{50}}^2 - 2432235/8192\, i\, a_{14}^3\, \overline{a_{50}} - 1398735/8192$
$i\, a_{14}^4)\, a_{50}^4 + (274365/8192\, i\, \overline{a_{14}}\, \overline{a_{50}}^4 - 11686799/36864\, i\, \overline{a_{14}}\, a_{14}^2\, \overline{a_{50}}^2$
$-1926355/3072\, i\, \overline{a_{14}}\, a_{14}^3\, \overline{a_{50}} - 27071951/73728\, i\, \overline{a_{14}}\, a_{14}^4)\, a_{50}^3 + (1398345/8192$
$i\, \overline{a_{14}}^2\, \overline{a_{50}}^4 + 11686799/36864\, i\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}}^3 - 14430251/36864\, i$
$\overline{a_{14}}^2\, a_{14}^3\, \overline{a_{50}} - 6445795/24576\, i\, \overline{a_{14}}^2\, a_{14}^4)\, a_{50}^2 + (2432235/8192\, i\, \overline{a_{14}}^3$
$\overline{a_{50}}^4 + 1926355/3072\, i\, \overline{a_{14}}^3\, a_{14}\, \overline{a_{50}}^3 + 14430251/36864\, i\, \overline{a_{14}}^3\, a_{14}^2$
$\overline{a_{50}}^2 - 4890577/73728\, i\, \overline{a_{14}}^3\, a_{14}^4)\, a_{50} + (1398735/8192\, i\, \overline{a_{14}}^4\, \overline{a_{50}}^4 +$
$27071951/73728\, i\, \overline{a_{14}}^4\, a_{14}\, \overline{a_{50}}^3 + 6445795/24576\, i\, \overline{a_{14}}^4\, a_{14}^2\, \overline{a_{50}}^2$
$+4890577/73728\, i\, \overline{a_{14}}^4\, a_{14}^3\, \overline{a_{50}}))),$

$$L_{17} = 0,$$

$L_{18} = ((87162075/65536\, \overline{a_{50}} + 110135025/65536\, a_{14})\, a_{50} + (87432345/65536\, \overline{a_{14}}\, \overline{a_{50}} +$
$110405295/65536\, \overline{a_{14}}\, a_{14}))\, a_{32}^7 + ((360548685/65536\, \overline{a_{50}} + 452689455/65536\, a_{14})\, a_{50} +$
$(364554231/65536\, \overline{a_{14}}\, \overline{a_{50}} + 456695001/65536\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}\, a_{32}^6 +$
$(((673678215/65536\, \overline{a_{50}} + 829293541/65536\, a_{14})\, a_{50} + (697710413/65536\, \overline{a_{14}}\, \overline{a_{50}} +$
$853325739/65536\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^2 + ((15836655285/4194304\, \overline{a_{50}}^2 + 28396962105/$
$2097152\, a_{14}\, \overline{a_{50}} + 46605578445/4194304\, a_{14}^2)\, a_{50}^2 + (12787931193/2097152\, \overline{a_{14}}\, \overline{a_{50}}^2$
$+21860089881/1048576\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}} + 35053889769/2097152\, \overline{a_{14}}\, a_{14}^2)\, a_{50}$
$+(9973050957/4194304\, \overline{a_{14}}^2\, \overline{a_{50}}^2 + 15481447017/2097152\, \overline{a_{14}}^2\, a_{14}\, \overline{a_{50}}$
$+23584815957/4194304\, \overline{a_{14}}^2\, a_{14}^2)))\, a_{32}^5 + (((824555025/65536\, \overline{a_{50}} + 967554603/65536$
$a_{14})\, a_{50} + (901436787/65536\, \overline{a_{14}}\, \overline{a_{50}} + 1044436365/65536\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^3$
$+((43707134025/4194304\, \overline{a_{50}}^2 + 76878846557/2097152\, a_{14}\, \overline{a_{50}} + 125497915713/$
$4194304\, a_{14}^2)\, a_{50}^2 + (39527069533/2097152\, \overline{a_{14}}\, \overline{a_{50}}^2 + 61985037117/1048576\, \overline{a_{14}}\, a_{14}$
$\overline{a_{50}} + 95605949005/2097152\, \overline{a_{14}}\, a_{14}^2)\, a_{50} + (37651677633/4194304\, \overline{a_{14}}^2\, \overline{a_{50}}^2$

$$+48686032589/2097152\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} + 66598919529/4194304\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}})\,a_{32}^4$$
$$+(((824555025/65536\,\overline{a_{50}} + 901436787/65536\,a_{14})\,a_{50} + (967554603/65536\,\overline{a_{14}}\,\overline{a_{50}}$$
$$+1044436365/65536\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^4 + ((28981128105/2097152\,\overline{a_{50}}^2 + 44999943165/$$
$$1048576\,a_{14}\,\overline{a_{50}} + 69356903073/2097152\,a_{14}^2)\,a_{50}^2 + (33979475709/1048576\,\overline{a_{14}}\,\overline{a_{50}}^2+$$
$$42170998941/524288\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} + 56318213613/1048576\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (43405983777/$$
$$2097152\,\overline{a_{14}}^2\,\overline{a_{50}}^2 + 42460517229/1048576\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} + 45088288713/2097152\,\overline{a_{14}}^2$$
$$a_{14}^2))\,\overline{a_{32}}^2 + ((3006188155/1572864\,\overline{a_{50}}^3 + 58891034945/4718592\,a_{14}\,\overline{a_{50}}^2+$$
$$13293213535/524288\,a_{14}^2\,\overline{a_{50}} + 75286740775/4718592\,a_{14}^3)\,a_{50}^3 + (7407275123/1572864$$
$$\overline{a_{14}}\,\overline{a_{50}}^3 + 123750853429/4718592\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2 + 78994830365/1572864\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}$$
$$+48624286793/1572864\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 + (2564523439/524288\,\overline{a_{14}}^2\,\overline{a_{50}}^3 + 30781782229/$$
$$1572864\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2 + 16733902393/524288\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} + 29087326339/1572864$$
$$\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (1180917287/524288\,\overline{a_{14}}^3\,\overline{a_{50}}^3 + 3204815059/524288\,\overline{a_{14}}^3$$
$$a_{14}\,\overline{a_{50}}^2 + 11324895691/1572864\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}} + 16783992437/4718592\,\overline{a_{14}}^3\,a_{14}^3)))$$
$$a_{32}^3 + (((673678215/65536\,\overline{a_{50}} + 697710413/65536\,a_{14})\,a_{50} + (829293541/65536\,\overline{a_{14}}$$
$$\overline{a_{50}} + 853325739/65536\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^5 + ((28981128105/2097152\,\overline{a_{50}}^2 + 33979475709/$$
$$1048576\,a_{14}\,\overline{a_{50}} + 43405983777/2097152\,a_{14}^2)\,a_{50}^2 + (44999943165/1048576\,\overline{a_{14}}\,\overline{a_{50}}^2$$
$$+42170998941/524288\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} + 42460517229/1048576\,\overline{a_{14}}\,a_{14}^2)$$
$$a_{50} + (69356903073/2097152\,\overline{a_{14}}^2\,\overline{a_{50}}^2 + 56318213613/1048576\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}+$$
$$45088288713/2097152\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^3 + ((1755405115/524288\,\overline{a_{50}}^3 + 89964038299/$$
$$4718592\,a_{14}\,\overline{a_{50}}^2 + 59246337287/1572864\,a_{14}^2\,\overline{a_{50}} + 112173023981/4718592\,a_{14}^3)\,a_{50}^3+$$
$$(58837068467/4718592\,\overline{a_{14}}\,\overline{a_{50}}^3 + 228379216735/4718592\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2 + 372971173997$$
$$/4718592\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} + 219479446369/4718592\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 + (31737893879/1572864$$
$$\overline{a_{14}}^2\,\overline{a_{50}}^3 + 249271941269/4718592\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2 + 94308566593/1572864$$
$$\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} + 137817067315/4718592\,\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (56602302757/4718592$$
$$\overline{a_{14}}^3\,\overline{a_{50}}^3 + 119005196785/4718592\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2 + 92125794571/4718592$$
$$\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}} + 10407170485/1572864\,\overline{a_{14}}^3\,a_{14}^3))\,\overline{a_{32}})\,a_{32}^2 + (((360548685/65536\,\overline{a_{50}}$$
$$+364554231/65536\,a_{14})\,a_{50} + (452689455/65536\,\overline{a_{14}}\,\overline{a_{50}} + 456695001/65536\,\overline{a_{14}}\,a_{14}))$$
$$\overline{a_{32}}^6 + ((43707134025/4194304\,\overline{a_{50}}^2 + 39527069533/2097152\,a_{14}\,\overline{a_{50}} + 37651677633/$$
$$4194304\,a_{14}^2)\,a_{50}^2 + (76878846557/2097152\,\overline{a_{14}}\,\overline{a_{50}}^2 + 61985037117/1048576\,\overline{a_{14}}$$
$$a_{14}\,\overline{a_{50}} + 48686032589/2097152\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (125497915713/4194304\,\overline{a_{14}}^2$$
$$\overline{a_{50}}^2 + 95605949005/2097152\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} + 66598919529/4194304\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^4$$
$$+((1755405115/524288\,\overline{a_{50}}^3 + 58837068467/4718592\,a_{14}\,\overline{a_{50}}^2 + 31737893879/$$
$$1572864\,a_{14}^2\,\overline{a_{50}} + 56602302757/4718592\,a_{14}^3)\,a_{50}^3 + (89964038299/4718592\,\overline{a_{14}}\,\overline{a_{50}}^3$$
$$+228379216735/4718592\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2 + 249271941269/4718592\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}$$

$+119005196785/4718592\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 + (59246337287/1572864\,\overline{a_{14}}^2\,\overline{a_{50}}^3 +$
$372971173997/4718592\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2 + 94308566593/1572864\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} +$
$92125794571/4718592\,\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (112173023981/4718592\,\overline{a_{14}}^3\,\overline{a_{50}}^3 +$
$219479446369/4718592\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2 + 137817067315/4718592\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}$
$+10407170485/1572864\,\overline{a_{14}}^3\,a_{14}^3))\,\overline{a_{32}}^2 + ((633940725/4194304\,\overline{a_{50}}^4 + 4568931835/$
$3145728\,a_{14}\,\overline{a_{50}}^3 + 99563050055/18874368\,a_{14}^2\,\overline{a_{50}}^2 + 8319667465/1048576\,a_{14}^3$
$\overline{a_{50}} + 156823728605/37748736\,a_{14}^4)\,a_{50}^4 + (2562585595/3145728\,\overline{a_{14}}\,\overline{a_{50}}^4 +$
$10572528617/2359296\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^3 + 62110952507/4718592\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}^2$
$+44026198577/2359296\,\overline{a_{14}}\,a_{14}^3\,\overline{a_{50}} + 90639697921/9437184\,\overline{a_{14}}\,a_{14}^4)$
$a_{50}^3 + (46367338055/18874368\,\overline{a_{14}}^2\,\overline{a_{50}}^4 + 12261535081/1572864\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^3$
$+132040657909/9437184\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}}^2 + 76058913611/4718592\,\overline{a_{14}}^2\,a_{14}^3\,\overline{a_{50}}$
$+16384130159/2097152\,\overline{a_{14}}^2\,a_{14}^4)\,a_{50}^2 + (3784781065/1048576\,\overline{a_{14}}^3\,\overline{a_{50}}^4 + 21228856369$
$/2359296\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^3 + 45957345611/4718592\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}^2 + 16522410697/2359296$
$\overline{a_{14}}^3\,a_{14}^3\,\overline{a_{50}} + 8695470763/3145728\,\overline{a_{14}}^3\,a_{14}^4)\,a_{50} + (72157242845/37748736\,\overline{a_{14}}^4$
$\overline{a_{50}}^4 + 41951672833/9437184\,\overline{a_{14}}^4\,a_{14}\,\overline{a_{50}}^3 + 71758643687/18874368\,\overline{a_{14}}^4\,a_{14}^2$
$\overline{a_{50}}^2 + 15692977153/9437184\,\overline{a_{14}}^4\,a_{14}^3\,\overline{a_{50}} + 15374548573/37748736\,\overline{a_{14}}^4\,a_{14}^4)))\,a_{32}$
$+(((87162075/65536\,\overline{a_{50}} + 87432345/65536\,a_{14})\,a_{50} + (110135025/65536\,\overline{a_{14}}\,\overline{a_{50}} +$
$110405295/65536\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^7 + ((15836655285/4194304\,\overline{a_{50}}^2 + 12787931193/$
$2097152\,a_{14}\,\overline{a_{50}} + 9973050957/4194304\,a_{14}^2)\,a_{50}^2 + (28396962105/2097152\,\overline{a_{14}}\,\overline{a_{50}}^2 +$
$21860089881/1048576\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} + 15481447017/2097152\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (46605578445$
$/4194304\,\overline{a_{14}}^2\,\overline{a_{50}}^2 + 35053889769/2097152\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} + 23584815957/4194304\,\overline{a_{14}}^2$
$a_{14}^2))\,\overline{a_{32}}^5 + ((3006188155/1572864\,\overline{a_{50}}^3 + 7407275123/1572864\,a_{14}\,\overline{a_{50}}^2 +$
$2564523439/524288\,a_{14}^2\,\overline{a_{50}} + 1180917287/524288\,a_{14}^3)\,a_{50}^3 + (58891034945/4718592$
$\overline{a_{14}}\,\overline{a_{50}}^3 + 123750853429/4718592\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2 + 30781782229/1572864\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}$
$+3204815059/524288\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 + (13293213535/524288\,\overline{a_{14}}^2\,\overline{a_{50}}^3 + 78994830365/$
$1572864\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2 + 16733902393/524288\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} + 11324895691/1572864$
$\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (75286740775/4718592\,\overline{a_{14}}^3\,\overline{a_{50}}^3 + 48624286793/1572864\,\overline{a_{14}}^3$
$a_{14}\,\overline{a_{50}}^2 + 29087326339/1572864\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}} + 16783992437/4718592\,\overline{a_{14}}^3$
$a_{14}^3))\,\overline{a_{32}}^3 + ((633940725/4194304\,\overline{a_{50}}^4 + 2562585595/3145728\,a_{14}\,\overline{a_{50}}^3 +$
$46367338055/18874368\,a_{14}^2\,\overline{a_{50}}^2 + 3784781065/1048576\,a_{14}^3\,\overline{a_{50}} + 72157242845/$
$37748736\,a_{14}^4)\,a_{50}^4 + (4568931835/3145728\,\overline{a_{14}}\,\overline{a_{50}}^4 + 10572528617/2359296\,\overline{a_{14}}\,a_{14}$
$\overline{a_{50}}^3 + 12261535081/1572864\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}^2 + 21228856369/2359296\,\overline{a_{14}}\,a_{14}^3\,\overline{a_{50}}$
$+41951672833/9437184\,\overline{a_{14}}\,a_{14}^4)\,a_{50}^3 + (99563050055/18874368\,\overline{a_{14}}^2\,\overline{a_{50}}^4$
$+62110952507/4718592\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^3 + 132040657909/9437184\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}}^2$

$$+45957345611/4718592\,\overline{a_{14}}^2\,a_{14}^3\,\overline{a_{50}}+71758643687/18874368\,\overline{a_{14}}^2\,a_{14}^4)\,a_{50}^2$$
$$+(8319667465/1048576\,\overline{a_{14}}^3\,\overline{a_{50}}^4+44026198577/2359296\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^3+$$
$$76058913611/4718592\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}^2+16522410697/2359296\,\overline{a_{14}}^3\,a_{14}^3\,\overline{a_{50}}$$
$$+15692977153/9437184\,\overline{a_{14}}^3\,a_{14}^4)\,a_{50}+(156823728605/37748736\,\overline{a_{14}}^4\,\overline{a_{50}}^4+$$
$$90639697921/9437184\,\overline{a_{14}}^4\,a_{14}\,\overline{a_{50}}^3+16384130159/2097152\,\overline{a_{14}}^4\,a_{14}^2\,\overline{a_{50}}^2$$
$$+8695470763/3145728\,\overline{a_{14}}^4\,a_{14}^3\,\overline{a_{50}}+15374548573/37748736\,\overline{a_{14}}^4\,a_{14}^4))\,\overline{a_{32}}),$$
$$L_{19}=0,$$
$$L_{20}=((57432375/8192\,i\,\overline{a_{50}}+1137161025/131072\,i\,a_{14})\,a_{50}+(916890975/131072\,i$$
$$\overline{a_{14}}\,\overline{a_{50}}+70945875/8192\,i\,\overline{a_{14}}\,a_{14}))\,a_{32}^8+((134808045/4096\,i\,\overline{a_{50}}+670186485/16384$$
$$i\,a_{14})\,a_{50}+(535156035/16384\,i\,\overline{a_{14}}\,\overline{a_{50}}+166527585/4096\,i\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}$$
$$a_{32}^7+(((259131425/4096\,i\,\overline{a_{50}}+2615401743/32768\,i\,a_{14})\,a_{50}+(2018477857/$$
$$32768\,i\,\overline{a_{14}}\,\overline{a_{50}}+320103525/4096\,i\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^2+((52239131955/$$
$$2097152\,i\,\overline{a_{50}}^2+363071095155/4194304\,i\,a_{14}\,\overline{a_{50}}+288766457235/4194304\,i\,a_{14}^2)\,a_{50}^2$$
$$+(159760060065/4194304\,i\,\overline{a_{14}}\,\overline{a_{50}}^2+4328420865/32768\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}$$
$$+439139882835/4194304\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50}+(54364985685/4194304\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2$$
$$+189710824665/4194304\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}+75017142585/2097152\,i\,\overline{a_{14}}^2\,a_{14}^2)))$$
$$a_{32}^6+(((228529385/4096\,i\,\overline{a_{50}}+1229818907/16384\,i\,a_{14})\,a_{50}+(813502653/16384\,i$$
$$\overline{a_{14}}\,\overline{a_{50}}+282301005/4096\,i\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^3+((40496732019/524288\,i$$
$$\overline{a_{50}}^2+593002857903/2097152\,i\,a_{14}\,\overline{a_{50}}+483399258021/2097152\,i\,a_{14}^2)\,a_{50}^2+$$
$$(225220928121/2097152\,i\,\overline{a_{14}}\,\overline{a_{50}}^2+3392515213/8192\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}+$$
$$720491776323/2097152\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50}+(58081264683/2097152\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2+$$
$$268270542613/2097152\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}+58758689913/524288\,i\,\overline{a_{14}}^2\,a_{14}^2))$$
$$\overline{a_{32}})\,a_{32}^5+((901436787/65536\,i\,a_{14}\,a_{50}-901436787/65536\,i\,\overline{a_{14}}\,\overline{a_{50}})$$
$$\overline{a_{32}}^4+((171160726527/2097152\,i\,\overline{a_{50}}^2+1486277484633/4194304\,i\,a_{14}\,\overline{a_{50}}+$$
$$1293895188501/4194304\,i\,a_{14}^2)\,a_{50}^2+(254880550923/4194304\,i\,\overline{a_{14}}\,\overline{a_{50}}^2+14454875749$$
$$/32768\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}+1814236195841/4194304\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50}+(-134604152205/$$
$$4194304\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2+296822821083/4194304\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}+$$
$$250190233629/2097152\,i\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^2+((7079673715/393216\,i\,\overline{a_{50}}^3$$
$$+13138242026747/113246208\,i\,a_{14}\,\overline{a_{50}}^2+4247151021631/18874368\,i\,a_{14}^2\,\overline{a_{50}}$$
$$+15285103244719/113246208\,i\,a_{14}^3)\,a_{50}^3+(3702757796741/113246208\,i\,\overline{a_{14}}\,\overline{a_{50}}^3$$
$$+807472654529/3538944\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2+50889666330365/113246208\,i\,\overline{a_{14}}$$
$$a_{14}^2\,\overline{a_{50}}+15256138942537/56623104\,i\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2+(210151550945/18874368$$
$$i\,\overline{a_{14}}^2\,\overline{a_{50}}^3+4769860800001/37748736\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^2+320183000069$$
$$/1179648\,i\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}}+6246723493553/37748736\,i\,\overline{a_{14}}^2\,a_{14}^3)\,a_{50}$$

$$+(-484141339823/113246208\,i\,\overline{a_{14}}^3\,\overline{a_{50}}^3 + 243069912349/18874368\,i\,\overline{a_{14}}^3\,a_{14}$$
$$\overline{a_{50}}^2 + 5248789305709/113246208\,i\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}} + 36458525719/1179648\,i$$
$$\overline{a_{14}}^3\,a_{14}^3)))\,a_{32}^4 + (((-228529385/4096\,i\,\overline{a_{50}} - 813502653/16384\,i\,a_{14})$$
$$a_{50} + (-1229818907/16384\,i\,\overline{a_{14}}\,\overline{a_{50}} - 282301005/4096\,i\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^5$$
$$+((174535548717/1048576\,i\,a_{14}\,\overline{a_{50}} + 202782697659/1048576\,i\,a_{14}^2)\,a_{50}^2+$$
$$(-174535548717/1048576\,i\,\overline{a_{14}}\,\overline{a_{50}}^2 + 215298132561/1048576\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50}+$$
$$(-202782697659/1048576\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2 - 215298132561/1048576\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}))$$
$$\overline{a_{32}}^3 + ((5683713139/196608\,i\,\overline{a_{50}}^3 + 6307215982555/28311552\,i\,a_{14}\,\overline{a_{50}}^2+$$
$$241458470847/524288\,i\,a_{14}^2\,\overline{a_{50}} + 8050511877743/28311552\,i\,a_{14}^3)\,a_{50}^3+$$
$$(542897712229/28311552\,i\,\overline{a_{14}}\,\overline{a_{50}}^3 + 658521222977/1769472\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2+$$
$$24759905249213/28311552\,i\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} + 7866790991713/14155776\,i\,\overline{a_{14}}\,a_{14}^3)$$
$$a_{50}^2 + (-111885926285/1572864\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^3 + 2026879505603/28311552\,i\,\overline{a_{14}}^2$$
$$a_{14}\,\overline{a_{50}}^2 + 262837725445/589824\,i\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} + 9144216646579/28311552\,i$$
$$\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (-1888982834671/28311552\,i\,\overline{a_{14}}^3\,\overline{a_{50}}^3 - 1242861286289$$
$$/14155776\,i\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2 + 735521308429/28311552\,i\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}$$
$$+29885212567/589824\,i\,\overline{a_{14}}^3\,a_{14}^3))\,\overline{a_{32}})\,a_{32}^3 + (((-259131425/4096\,i\,\overline{a_{50}}$$
$$-2018477857/32768\,i\,a_{14})\,a_{50} + (-2615401743/32768\,i\,\overline{a_{14}}\,\overline{a_{50}} - 320103525/4096\,i$$
$$\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^6 + ((-171160726527/2097152\,i\,\overline{a_{50}}^2 - 254880550923/4194304\,i\,a_{14}$$
$$\overline{a_{50}} + 134604152205/4194304\,i\,a_{14}^2)\,a_{50}^2 + (-1486277484633/4194304\,i\,\overline{a_{14}}$$
$$\overline{a_{50}}^2 - 14454875749/32768\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} - 296822821083/4194304\,i$$
$$\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-1293895188501/4194304\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2 - 1814236195841/4194304$$
$$i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} - 250190233629/2097152\,i\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^4+$$
$$((2342782608571/18874368\,i\,a_{14}\,\overline{a_{50}}^2 + 3086984698253/9437184\,i\,a_{14}^2\,\overline{a_{50}}+$$
$$1369010720869/6291456\,i\,a_{14}^3)\,a_{50}^3 + (-2342782608571/18874368\,i\,\overline{a_{14}}\,\overline{a_{50}}^3+$$
$$27968334515063/56623104\,i\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} + 11260102543723/28311552\,i\,\overline{a_{14}}\,a_{14}^3)$$
$$a_{50}^2 + (-3086984698253/9437184\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^3 - 27968334515063/56623104\,i\,\overline{a_{14}}^2$$
$$a_{14}\,\overline{a_{50}}^2 + 10357744416761/56623104\,i\,\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (-1369010720869/$$
$$6291456\,i\,\overline{a_{14}}^3\,\overline{a_{50}}^3 - 11260102543723/28311552\,i\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2-$$
$$10357744416761/56623104\,i\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}))\,\overline{a_{32}}^2 + ((2377891303/1048576\,i\,\overline{a_{50}}^4$$
$$+502194011213/18874368\,i\,a_{14}\,\overline{a_{50}}^3 + 5300815187543/56623104\,i\,a_{14}^2\,\overline{a_{50}}^2+$$
$$828890694257/6291456\,i\,a_{14}^3\,\overline{a_{50}} + 3665244498607/56623104\,i\,a_{14}^4)\,a_{50}^4+$$
$$1(26542062895/8874368\,i\,\overline{a_{14}}\,\overline{a_{50}}^4 + 65796014287/1179648\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^3+$$
$$6276809583005/28311552\,i\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}^2 + 2256202464011/7077888\,i\,\overline{a_{14}}\,a_{14}^3\,\overline{a_{50}}$$
$$+8878791608033/56623104\,i\,\overline{a_{14}}\,a_{14}^4)\,a_{50}^3 + (-822424942739/56623104\,i$$

$$\overline{a_{14}}^2\,\overline{a_{50}}^4 + 320245203031/28311552\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^3 + 571901027869/$$
$$3538944\,i\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}}^2 + 7487918919245/28311552\,i\,\overline{a_{14}}^2\,a_{14}^3\,\overline{a_{50}}$$
$$+838065382067/6291456\,i\,\overline{a_{14}}^2\,a_{14}^4)\,a_{50}^2 + (-180032369557/6291456\,i\,\overline{a_{14}}^3$$
$$\overline{a_{50}}^4 - 362088460819/7077888\,i\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^3 + 123400700437/9437184\,i\,\overline{a_{14}}^3$$
$$a_{14}^2\,\overline{a_{50}}^2 + 283562357767/3538944\,i\,\overline{a_{14}}^3\,a_{14}^3\,\overline{a_{50}} + 869620747613/18874368$$
$$i\,\overline{a_{14}}^3\,a_{14}^4)\,a_{50} + (-882520337335/56623104\,i\,\overline{a_{14}}^4\,\overline{a_{50}}^4 - 72937510247/$$
$$2097152\,i\,\overline{a_{14}}^4\,a_{14}\,\overline{a_{50}}^3 - 1225078441135/56623104\,i\,\overline{a_{14}}^4\,a_{14}^2\,\overline{a_{50}}^2 + 14249871181/$$
$$6291456\,i\,\overline{a_{14}}^4\,a_{14}^3\,\overline{a_{50}} + 15406069847/3145728\,i\,\overline{a_{14}}^4\,a_{14}^4)))\,a_{32}^2 + (((-134808045/$$
$$4096\,i\,\overline{a_{50}} - 535156035/16384\,i\,a_{14})\,a_{50} + (-670186485/16384\,i\,\overline{a_{14}}$$
$$\overline{a_{50}} - 166527585/4096\,i\,\overline{a_{14}}\,a_{14}))\,\overline{a_{32}}^7 + ((-40496732019/524288\,i\,\overline{a_{50}}^2$$
$$-225220928121/2097152\,i\,a_{14}\,\overline{a_{50}} - 58081264683/2097152\,i\,a_{14}^2)\,a_{50}^2 +$$
$$(-593002857903/2097152\,i\,\overline{a_{14}}\,\overline{a_{50}}^2 - 3392515213/8192\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}} -$$
$$268270542613/2097152\,i\,\overline{a_{14}}\,a_{14}^2)\,a_{50} + (-483399258021/2097152\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^2 -$$
$$720491776323/2097152\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}} - 58758689913/524288\,i\,\overline{a_{14}}^2\,a_{14}^2))\,\overline{a_{32}}^5$$
$$+((-5683713139/196608\,i\,\overline{a_{50}}^3 - 542897712229/28311552\,i\,a_{14}\,\overline{a_{50}}^2 +$$
$$111885926285/1572864\,i\,a_{14}^2\,\overline{a_{50}} + 1888982834671/28311552\,i\,a_{14}^3)\,a_{50}^3 +$$
$$(-6307215982555/28311552\,i\,\overline{a_{14}}\,\overline{a_{50}}^3 - 658521222977/1769472\,i\,\overline{a_{14}}\,a_{14}\,\overline{a_{50}}^2 -$$
$$2026879505603/28311552\,i\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}} + 1242861286289/14155776\,i\,\overline{a_{14}}\,a_{14}^3)\,a_{50}^2 +$$
$$(-241458470847/524288\,i\,\overline{a_{14}}^2\,\overline{a_{50}}^3 - 24759905249213/28311552\,i\,\overline{a_{14}}^2$$
$$a_{14}\,\overline{a_{50}}^2 - 262837725445/589824\,i\,\overline{a_{14}}^2\,a_{14}^2\,\overline{a_{50}} - 735521308429/28311552\,i$$
$$\overline{a_{14}}^2\,a_{14}^3)\,a_{50} + (-8050511877743/28311552\,i\,\overline{a_{14}}^3\,\overline{a_{50}}^3 - 7866790991713/14155776$$
$$i\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^2 - 9144216646579/28311552\,i\,\overline{a_{14}}^3\,a_{14}^2$$
$$\overline{a_{50}} - 29885212567/589824\,i\,\overline{a_{14}}^3\,a_{14}^3))\,\overline{a_{32}}^3 + ((182888201231/9437184\,i\,a_{14}$$
$$\overline{a_{50}}^3 + 2393495170037/28311552\,i\,a_{14}^2\,\overline{a_{50}}^2 + 133266454145/1048576\,i\,a_{14}^3\,\overline{a_{50}}$$
$$+1821216490675/28311552\,i\,a_{14}^4)\,a_{50}^4 + (-182888201231/9437184\,i\,\overline{a_{14}}\,\overline{a_{50}}^4 +$$
$$2331572165411/14155776\,i\,\overline{a_{14}}\,a_{14}^2\,\overline{a_{50}}^2 + 345126487813/1179648\,i\,\overline{a_{14}}\,a_{14}^3$$
$$\overline{a_{50}} + 4328664416479/28311552\,i\,\overline{a_{14}}\,a_{14}^4)\,a_{50}^3 + (-2393495170037/28311552\,i\,\overline{a_{14}}^2$$
$$\overline{a_{50}}^4 - 2331572165411/14155776\,i\,\overline{a_{14}}^2\,a_{14}\,\overline{a_{50}}^3 + 2809589627527/14155776\,i$$
$$\overline{a_{14}}^2\,a_{14}^3\,\overline{a_{50}} + 3481564759805/28311552\,i\,\overline{a_{14}}^2\,a_{14}^4)\,a_{50}^2 + (-133266454145/$$
$$1048576\,i\,\overline{a_{14}}^3\,\overline{a_{50}}^4 - 345126487813/1179648\,i\,\overline{a_{14}}^3\,a_{14}\,\overline{a_{50}}^3 -$$
$$2809589627527/14155776\,i\,\overline{a_{14}}^3\,a_{14}^2\,\overline{a_{50}}^2 + 326464833595/9437184\,i\,\overline{a_{14}}^3\,a_{14}^4)$$
$$a_{50} + (-1821216490675/28311552\,i\,\overline{a_{14}}^4\,\overline{a_{50}}^4 - 4328664416479/28311552\,i\,\overline{a_{14}}^4$$
$$a_{14}\,\overline{a_{50}}^3 - 3481564759805/28311552\,i\,\overline{a_{14}}^4\,a_{14}^2\,\overline{a_{50}}^2 - 326464833595/9437184$$
$$i\,\overline{a_{14}}^4\,a_{14}^3\,\overline{a_{50}}))\,\overline{a_{32}})\,a_{32} + (((-57432375/8192\,i\,\overline{a_{50}} - 916890975/$$

$$131072\, i\, a_{14})\, a_{50} + (-1137161025/131072\, i\, \overline{a_{14}}\, \overline{a_{50}} - 70945875/8192$$
$$i\, \overline{a_{14}}\, a_{14}))\, \overline{a_{32}}^{8} + ((-52239131955/2097152\, i\, \overline{a_{50}}^{2} - 159760060065/4194304\, i\, a_{14}$$
$$\overline{a_{50}} - 54364985685/4194304\, i\, a_{14}^{2})\, a_{50}^{2} + (-363071095155/4194304\, i\, \overline{a_{14}}\, \overline{a_{50}}^{2}$$
$$-4328420865/32768\, i\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}} - 189710824665/4194304\, i\, \overline{a_{14}}\, a_{14}^{2})\, a_{50}$$
$$+(-288766457235/4194304\, i\, \overline{a_{14}}^{2}\, \overline{a_{50}}^{2} - 439139882835/4194304\, i\, \overline{a_{14}}^{2}\, a_{14}\, \overline{a_{50}}$$
$$-75017142585/2097152\, i\, \overline{a_{14}}^{2}\, a_{14}^{2}))\, \overline{a_{32}}^{6} + ((-7079673715/393216\, i\, \overline{a_{50}}^{3} -$$
$$3702757796741/113246208\, i\, a_{14}\, \overline{a_{50}}^{2} - 210151550945/18874368\, i\, a_{14}^{2}\, \overline{a_{50}} +$$
$$484141339823/113246208\, i\, a_{14}^{3})\, a_{50}^{3} + (-13138242026747/113246208\, i\, \overline{a_{14}}\, \overline{a_{50}}^{3} -$$
$$807472654529/3538944\, i\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}}^{2} - 4769860800001/37748736\, i\, \overline{a_{14}}\, a_{14}^{2}$$
$$\overline{a_{50}} - 243069912349/18874368\, i\, \overline{a_{14}}\, a_{14}^{3})\, a_{50}^{2} + (-4247151021631/18874368\, i\, \overline{a_{14}}^{2}$$
$$\overline{a_{50}}^{3} - 50889666330365/113246208\, i\, \overline{a_{14}}^{2}\, a_{14}\, \overline{a_{50}}^{2} - 320183000069/1179648\, i$$
$$\overline{a_{14}}^{2}\, a_{14}^{2}\, \overline{a_{50}} - 5248789305709/113246208\, i\, \overline{a_{14}}^{2}\, a_{14}^{3})\, a_{50} +$$
$$(-15285103244719/113246208\, i\, \overline{a_{14}}^{3}\, \overline{a_{50}}^{3} - 15256138942537/56623104\, i\, \overline{a_{14}}^{3}\, a_{14}$$
$$\overline{a_{50}}^{2} - 6246723493553/37748736\, i\, \overline{a_{14}}^{3}\, a_{14}^{2}\, \overline{a_{50}} - 36458525719/1179648\, i\, \overline{a_{14}}^{3}$$
$$a_{14}^{3}))\, \overline{a_{32}}^{4} + ((-2377891303/1048576\, i\, \overline{a_{50}}^{4} - 26542062895/18874368\, i\, a_{14}$$
$$\overline{a_{50}}^{3} + 822424942739/56623104\, i\, a_{14}^{2}\, \overline{a_{50}}^{2} + 180032369557/6291456\, i\, a_{14}^{3}\, \overline{a_{50}}$$
$$+882520337335/56623104\, i\, a_{14}^{4})\, a_{50}^{4} + (-502194011213/18874368\, i\, \overline{a_{14}}\, \overline{a_{50}}^{4} -$$
$$65796014287/1179648\, i\, \overline{a_{14}}\, a_{14}\, \overline{a_{50}}^{3} - 320245203031/28311552\, i\, \overline{a_{14}}\, a_{14}^{2}$$
$$\overline{a_{50}}^{2} + 362088460819/7077888\, i\, \overline{a_{14}}\, a_{14}^{3}\, \overline{a_{50}} + 72937510247/2097152\, i\, \overline{a_{14}}$$
$$a_{14}^{4})\, a_{50}^{3} + (-5300815187543/56623104\, i\, \overline{a_{14}}^{2}\, \overline{a_{50}}^{4} - 6276809583005/28311552\, i$$
$$\overline{a_{14}}^{2}\, a_{14}\, \overline{a_{50}}^{3} - 571901027869/3538944\, i\, \overline{a_{14}}^{2}\, a_{14}^{2}\, \overline{a_{50}}^{2} -$$
$$123400700437/9437184\, i\, \overline{a_{14}}^{2}\, a_{14}^{3}\, \overline{a_{50}} + 1225078441135/56623104\, i\, \overline{a_{14}}^{2}\, a_{14}^{4})$$
$$a_{50}^{2} + (-828890694257/6291456\, i\, \overline{a_{14}}^{3}\, \overline{a_{50}}^{4} - 2256202464011/7077888\, i\, \overline{a_{14}}^{3}$$
$$a_{14}\, \overline{a_{50}}^{3} - 7487918919245/28311552\, i\, \overline{a_{14}}^{3}\, a_{14}^{2}\, \overline{a_{50}}^{2} -$$
$$283562357767/3538944\, i\, \overline{a_{14}}^{3}\, a_{14}^{3}\, \overline{a_{50}} - 14249871181/6291456\, i\, \overline{a_{14}}^{3}\, a_{14}^{4})$$
$$a_{50} + (-3665244498607/56623104\, i\, \overline{a_{14}}^{4}\, \overline{a_{50}}^{4} - 8878791608033/56623104\, i\, \overline{a_{14}}^{4}$$
$$a_{14}\, \overline{a_{50}}^{3} - 838065382067/6291456\, i\, \overline{a_{14}}^{4}\, a_{14}^{2}\, \overline{a_{50}}^{2} - 869620747613/18874368$$
$$i\, \overline{a_{14}}^{4}\, a_{14}^{3}\, \overline{a_{50}} - 15406069847/3145728\, i\, \overline{a_{14}}^{4}\, a_{14}^{4}))\, \overline{a_{32}}^{2} +$$
$$((1887928965/4194304\, i\, a_{14}\, \overline{a_{50}}^{4} + 3066210355/1048576\, i\, a_{14}^{2}\, \overline{a_{50}}^{3} +$$
$$46572197425/6291456\, i\, a_{14}^{3}\, \overline{a_{50}}^{2} + 8817041415/1048576\, i\, a_{14}^{4}\, \overline{a_{50}} +$$
$$447761683655/12582912\, i\, a_{14}^{5})\, a_{50}^{5} + (-1887928965/4194304\, i\, \overline{a_{14}}\, \overline{a_{50}}^{5} +$$
$$130696810133/18874368\, i\, \overline{a_{14}}\, a_{14}^{2}\, \overline{a_{50}}^{3} + 63664785689/3145728\, i\, \overline{a_{14}}\, a_{14}^{3}$$
$$\overline{a_{50}}^{2} + 887817690043/37748736\, i\, \overline{a_{14}}\, a_{14}^{4}\, \overline{a_{50}} + 31357497281/3145728\, i\, \overline{a_{14}}$$
$$a_{14}^{5})\, a_{50}^{4} + (-3066210355/1048576\, i\, \overline{a_{14}}^{2}\, \overline{a_{50}}^{5} - 130696810133/18874368\, i$$

$$\overline{a_{14}}^2\, a_{14}\, \overline{a_{50}}^4 + 169125140041/9437184\, i\, \overline{a_{14}}^2\, a_{14}^3\, \overline{a_{50}}^2 +$$
$$76228212917/3145728\, i\, \overline{a_{14}}^2\, a_{14}^4\, \overline{a_{50}} + 198840646315/18874368\, i\, \overline{a_{14}}^2\, a_{14}^5)$$
$$a_{50}^3 + (-46572197425/6291456\, i\, \overline{a_{14}}^3\, \overline{a_{50}}^5 - 63664785689/3145728\, i\, \overline{a_{14}}^3$$
$$a_{14}\, \overline{a_{50}}^4 - 169125140041/9437184\, i\, \overline{a_{14}}^3\, a_{14}^2\, \overline{a_{50}}^3 + 190622241725/18874368$$
$$i\, \overline{a_{14}}^3\, a_{14}^4\, \overline{a_{50}} + 16123479517/3145728\, i\, \overline{a_{14}}^3\, a_{14}^5)\, a_{50}^2 +$$
$$(-8817041415/1048576\, i\, \overline{a_{14}}^4\, \overline{a_{50}}^5 - 887817690043/37748736\, i\, \overline{a_{14}}^4\, a_{14}\, \overline{a_{50}}^4$$
$$-76228212917/3145728\, i\, \overline{a_{14}}^4\, a_{14}^2\, \overline{a_{50}}^3 - 190622241725/18874368\, i\, \overline{a_{14}}^4$$
$$a_{14}^3\, \overline{a_{50}}^2 + 4209879317/4194304\, i\, \overline{a_{14}}^4\, a_{14}^5)\, a_{50} + (-44761683655/12582912$$
$$i\, \overline{a_{14}}^5\, \overline{a_{50}}^5 - 31357497281/3145728\, i\, \overline{a_{14}}^5\, a_{14}\, \overline{a_{50}}^4 -$$
$$198840646315/18874368\, i\, \overline{a_{14}}^5\, a_{14}^2\, \overline{a_{50}}^3 - 16123479517/3145728\, i\, \overline{a_{14}}^5\, a_{14}^3$$
$$\overline{a_{50}}^2 - 4209879317/4194304\, i\, \overline{a_{14}}^5\, a_{14}^4\, \overline{a_{50}}))).$$