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AREA: COMPUTATION AND MATHEMATICS

Protection of Graphs

Generic results with emphasis on Cartesian and Lexicographic product of graphs

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FINAL PROJECT SHEET

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Abstract

Suppose that one or more entities are stationed at some of the vertices of a graph G and that an entity at a vertex can deal with a problem at any vertex in its closed neighbourhood. Informally, we say that G is protected under a given placement of entities if there exists at least one entity available to handle a problem at any vertex.

Cockayne et al. [Bulletin of the Institute of Combinatorics and its Applications 39 (2003) 87–100] proposed four properties of such functions under which the entire graph may be protected according to a certain strategy. In each case the parameter of interest will be the minimum weight of a function in the subclass (minimum number of entities used).

In this work, we obtain closed formulae and tight bounds for two of these protection types: weak Roman domination number and secure domination number; focusing in lexicographic and Cartesian product graphs in terms of invariants of the factor graphs involved in the product.

It is shown that the problem of computing the weak Roman domination number (Henning and Hedetniemi [Discrete Math. 266 (2003) 239-251]) and secure domination number (Boumediene Merouane and Chellali [Inform. Process. Lett. 115 (10) (2015) 786–790.]) is NP-Hard, even when restricted to bipartite or chordal graphs. This suggests finding the domination number for special classes of graphs or obtaining good bounds on this invariant.

Both approaches followed in this work, M. Valveny, H. Pérez-Rosés and J. A. Rodríguez-Velázquez [Discrete Math. 263 (2019) 257-270] and M. Valveny and J. A. Rodríguez-Velázquez [Filomat 33 (1) (2019) 319-333], have been published in Discrete Applied Mathematics and Filomat respectively.

Keywords: Graphs, Domination, Protection of graphs, Weak Roman domination, Secure domination, Product of graphs, Lexicographic product of graphs, Cartesian product of graphs.

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Chapter 1

Introduction

Any real world situation can be illustrated diagrammatically with set of points joined with lines. A mathematical abstraction of situations which focuses on the way in which the points are connected together give rise to the concept called graph [1, 2].

1.1 Context and justification of the Work

Understanding the relations and the effects on the interconnection between these points can bring theoretical knowledge which contributes to a number of applications in communication, molecular physics and chemistry, social networks, biological sciences, computational linguistics, and in other numerous areas. In graph theory, one of the extensively researched branches is domination in graphs [3]. As how its shown in [8], now we will illustrate some examples of the applications of this area of study:

- **Social Network Theory** If we consider that the online social network can be represented as a graph of relationship with individuals representing the nodes and the social interactions as edges, then domination plays a vital role in analyzing the real effect on a real online social network data through simulation.

  The domination concept can be applied to the social network graph to determinate the amount of influence that is possessed by an individual as well as its impact to their related neighbours. Persons can play different roles as they are affected by their peers. So, each one can spread influence throughout the entire community in the social network[4]. A good example nowadays is the prediction of the extension of COVID-19.

- **Computer Communication Networks** Domination plays an important role in computer and communication networks to route the network.

- **Mobile Ad-hoc Network** A Mobile Ad-hoc Network (MANET) is a self-configurable infrastructureless network connecting the mobile devices in wireless mode [5]. Domination has been commonly used for routing and broadcasting the information in mobile ad-hoc networks. Domination is used in this field to reduce the communication and the storage overhead by keeping its size to be minimal.

- **Wireless Sensor Network** A Wireless Sensor Network is a type of wireless network which consists of spatially distributed autonomous sensors to monitor the physical or environmental conditions and to broadcast their information through the wireless network to a main location[6, 7].

  The main activity here is to route the information between nodes in time. This can be very challenging because of the inherit characteristics of distinguishing this networks from other networks. Again, the domination is needed to reduce the routing overheads.

1.2 Aims of the Work

Observing our surrounding thoroughly, we can easily realize that any daily thing can be schematized by representing it as a graph. In particular, referring to computer science, graphs are easily associated to the structure of any network. Finding new working techniques for graphs has direct repercussions on how networks are used remarking the importance of understanding graphs in our environment.

The fact of contributing to scientific world providing new ideas and research, has been the main reason that has motivated the election of this work. The area of domination has been chosen as it is easily associated to networks and its key nodes. Finally, the research is restricted to lexicographic and Cartesian product graphs as main graphs families are already deepen studied and with this limited range results can be more accurate.

As the work is strictly associated to theory, it is also important the fact of contributing with new knowledge and to find new formulae which simplify previous methods. Creating new theory and learning how to work with it can imply huge progress in our daily routines demonstrating the importance of the research work and the impact of theoretical investigation in real implementation.

Finally, an important objective is to deepen the skills achieved during the years of work and study so that they are firmly consolidated. Realizing such a research work also implies to extend the current knowledge in order to adapt it to the needs of the project.

1.3 Approach and method followed

A network is a graph on which a set of additional attributes has been defined. For the aim of the work, we focus only on those attributes related to protection pf graphs and do not depend on additional attributes.

The following approach to protection of a graph was described by Cockayne et al. [19]. Suppose that one or more entities are stationed at some of the vertices of a graph G and that an entity at a vertex can deal with a problem at any vertex in its closed neighborhood. Informally, we say that G is protected under a given placement of entities if there exists at least one entity available to handle a problem at any vertex. The domination number of a graph is the minimum amount of entities needed in order to consider the graph as protected according to the type of domination.

It was shown in [28] that the problem of computing $\gamma(G)$ is NP-hard, even when restricted to bipartite or chordal graphs, also the problem of computing $\gamma_s(G)$ is also NP-hard, even when restricted to split graphs [14]. This suggests finding the weak Roman domination number and the secure domination number for special classes of graphs or obtaining good bounds on these invariants of the factor graphs.

The intention of this article is to research on the domination number of graphs. For doing so, we will follow two of the approaches described in detail in Section 2.3. On the one hand, we will have an overview on weak Roman domination focusing on lexicographic product graphs. On the other hand, we will study secure domination with a slight emphasis on Cartesian product.

In weak Roman domination, we will remind some remarks and show only a few results on common classes of graphs. The effort for this approach is directed mainly thowards the study of families obtained by lexicographic product graphs. This strategy is followed as these families of graphs are not deeply studied for weak Roman domination number and so more results can be achieved. We cite the following works on domination theory for lexicographic product: the domination number was studied in [31, 35], the Roman domination number was studied in [37], the rainbow domination number was studied in [38], the super domination number was studied in [21], while the doubly connected domination number was studied in [9].

The Cartesian product is a straightforward and natural construction, and is in many respects the simplest graph product [25, 30]. Hypercubes, Hamming graphs, grid graphs, cylinder graphs and torus graphs are some particular cases of this product. This product has been extensively investigated from various perspectives. For instance, the most popular open problem in the area of domination theory is known as Vizing's conjecture. Vizing [39] suggested that for any graphs G and H,

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H).$$

Several researchers have worked on it, for instance, some partial results appear in [15, 25]. On the contrary, secure domination theory is less developed which allows us to do more research on general bounds. For this reason, the approach followed will be the opposite than the previous case. We will present more results on general families and relations with other graph properties and have a smaller dedication on the product.

Weak Roman domination and Lexicographic product graphs was already presented as final degree thesis. Secure domination and Cartesian product graphs will be presented in the current document as an extension of the work which offers more results in the same scientific field. As a result of this work we have published the following two papers: On the weak Roman domination number of lexicographic product graphs [41] and Protection of graphs with emphasis on Cartesian product graphs [42].

1.4 Planning of the Work

This thesis is presented in a unique submission which means that the work needs to be structured and self-organized. As it is a research work, the result obtained may seem abstract and can give the impression that all the work can be done disorderly and in the end. By contrary, continuity and temporary planning are fundamental for progressing and achieving objectives.

First of all, it is needed to define each of the parts and identify all the dependencies between the tasks. In our case, the first need is to research on the topic and search for all possible related information. When doing research work, it is very important to keep informed about the subject of study which means investing a lot of time on researching related works. In this way it is possible to see already known results and the strategies used for achieving them. Hence, this is the first step which needs to be completed once the proposal of the thesis is done.

Next, it is possible to start thinking in new ideas and try to apply them in order to see if they may be in the good direction. This part includes the search of examples and contrast with other results. Finally, only by formalization of the work, real results (*e.g. Theorems, Corollaries, etc.*) can be defined.

These three steps can be repeated as many times as needed and always following the same order. In this way, it is possible to do incremental work on the already known results which will enrich the value of the work. Thus, it is interesting to keep informed about related work during the realization of the thesis and review already finished results.

1.5 Summary of products obtained

In this work we obtain a set of bounds and closed formulae for both weak Roman domination number and secure domination number. All of the results contribute with the investigation, but some of them do it beyond the others. We show properties of the graphs studied and define bounds and families of graphs that improve the results in a much general and useful way. For this reason we consider important to remark some of them.

Starting with weak Roman domination, Theorem 19 shows that $\gamma_r(G \circ H) \leq \gamma(G)\gamma_r(H)$. This inequality makes us realize that the first graph of the product is the one on which the entities needed truly depend. Again, the importance of the first graph of the product is shown in Theorem 23 where three lower bounds are given only considering this graph. This thought is corroborated in Lemma 65 where it is shown that when operating $G \circ H$, if maintaining $\gamma(H) \geq 4$, then the second graph does not change the result for any graph H. It is true that graph H is involved in the final graph, but if it is a relatively big and with low density it will not improve more than the stated result where $\gamma_r(G \circ H) \leq 2\gamma_t(G)$.

Also Theorem 37 must be remarked as it gives a formula to expand (or reduce) the first graph knowing exactly the amount of entities to protect the generated graph according to the previous one. This result is important because is the only one in the work that gives an exact relation of entities between two similar graphs. In the same way, Lemma 65 is another important result as it reduces the cost of calculating the entities needed to protect a lexicographic product graph by dismissing great part of the nodes.

Referring to the study of secure domination number, we obtain some good bounds as shown in Theorem 53 for general graphs and Theorem 61 for Cartesian product graphs. In both cases we obtain some chains of inequalities which help accurating the precision of the results. Apart from the general bounds, we derive new inequalities of Nordhaus-Gaddum type involving secure domination and weak Roman domination in Theorem 52. Also in Theorem 64 it is shown that secure domination is not following a Vizing-like conjecture.

Theorem 48 must be remarked as with the consideration of twin nodes, the bound for calculating both weak Roman domination number and secure domination of any graph is decreased as redundant vertices are discarded. The relevance of this result remains on the fact that that the bound can be extended to the most types of protection on graphs.

Finally, we also provide some bounds for known families of graphs in the Cartesian product like complete graphs in Propositions 57-60 and star graphs in Proposition 63. For the case of tree graphs, we calculate the amount of entities needed providing a reduction method 41. This result means an improvement as for this family of graphs the cost of computing the secure domination number is reduced considerably.

1.6 Description of chaptes

In this document we research on protection of graphs, where we obtain results and bounds for different classes and families. First, in Chapter 2, we introduce the concepts and notation needed to follow the thesis. In this section we present an overview on the bases of graph theory 2.1, product composition 2.2 and protection of graphs 2.3.

The results of the study are gathered between Chapters 3-5. In Chapter 3 we show common bounds for both weak Roman domination and secure domination while in Chapter 4 and 5 we consider the strategies separately.

Chapter 4 corresponds to the work presented in final degree thesis where we explore particular results for weak Roman domination. In Section 4.1, we focus on the study of weak Roman domination in lexicographic product graphs comparing it with other known metrics. The subsection is divided in three subparts according to the nature of the result. First, we have an overview on upper bounds of the domination in Section 4.1.1 followed by the lower bounds in Section 4.1.2. Finally, we show in Section 4.1.3 closed formulae for the product. Notice that the demonstration of Theorem 37 is moved to Annex 1 due to its huge extension.

Section 5 is devoted to obtain general bounds on $\gamma_r(G)$ and $\gamma_s(G)$ in terms of several invariants of G. As a consequence of the study we derive new inequalities of Nordhaus-Gaddum type involving secure domination and weak Roman domination. Later, in Subsection 5.1 the study is restricted to the particular case of Cartesian product graphs.

After the collection of results, we give some conclusions obtained from the realization of this work in Chapter 6. In this Chapter we treat the lessons learned from the realization of the work in Section 6.1. Next, we have an overview of the achieved objectives in Section 6.2 according to the proposed aims in Section 1.2. After this, we present an overview of the suitability of the planning of the work followed in Section 6.3. Derived from the research done, there are still some open problems left which are covered in Section 6.4. These problems represent future study fields which would complement the bounds and formulae obtained giving more precision to the results obtained.

For complementary terms and new definitions of the work, we have Chapters Glossary and Acronyms which provide the needed descriptions for the referred terminology. Finally, in the end of the document we provide the mentioned Annex 1 for the proof of Theorem 37.

Chapter 2

Bases on graph theory and domination

In this Chapter we present a reminder of the theory needed to follow the thesis. First, we start in Section 2.1 showing an overview on the bases of graph theory defining the most used notation 2.1.1 and graph families 2.1.2. Next, in Section 2.2 where we define graph operation having Subsection 2.2.1 for Cartesian product and Subsection 2.2.2 for lexicographic product. Finally, in Section 2.3, we define protection of graphs and the domination strategies used.

2.1 Basis on graph theory

The configurations of nodes and connections between them appear with frequency in different contexts to represent "networks" of different types. Formally these structures are combinatory structures called graphs.

2.1.1 Graph notation

Definition 1. *A graph $G = (V, E)$ is an orderly pair where V is a nonempty finite set and E is a set of non-ordered pairs u, v of elements of V with $u \neq v$. The elements of V are called **vertices**, or **nodes**, of G and the elements of E are called **edges** of G. The **order** of G is the number of vertices and the **size** of G is the number of edges.*

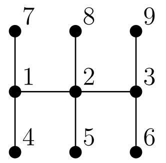

Figure 2.1: Example of graph definition.

Figure 2.1 shows a graphic representation of a graph $G = (V, E)$ where V is the set of vertices and E the set of edges. In this example the order of G is $n = 9$ and the size is $m = 8$. V and E are defined as following:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$E = \{\{1, 2\}, \{1, 4\}, \{1, 7\}, \{2, 5\}, \{2, 8\}, \{2, 3\}, \{3, 6\}, \{3, 9\}\}.$$

Definition 2. *Two vertices $u, v \in V$ of a graph $G = (V, E)$ are **adjacent**, $u \sim v$, if and only if the edge $\{u, v\}$ exists; $\{u, v\} \in E$. Another common denomination for referring to an edge is that vertices u and v are **neighbour** vertices.*

From now on, to avoid confusion, an edge $\{u, v\}$ also will be denoted by uv and, in this case, will be written as $uv \in E$ instead of $\{u, v\} \in E$.

Definition 3. *Given a vertex $v \in V$ of a graph $G = (V, E)$ the **degree** of v, $\delta(v)$, is defined as the number of edges that are incident to v. That is:*

$$\delta(v) = |\{u \in V : v \sim u\}| = |\{u \in V : uv \in E\}|$$

*The **maximum degree** of a graph G is denoted by $\Delta(G)$ and the **minimum degree** of a graph $\delta(G)$. The vertices of degree zero are called **isolated vertices**.*

Let us consider the graph of Figure 2.1 as G. In this case, $\Delta(G) = 4$, $\delta(G) = 1$, $\delta(1) = \delta(3) = 3$, $\delta(2) = 4$ and all the other vertices have degree one.

Definition 4. *A **walk** in a graph $G = (V, E)$ is a sequence of vertices $v_1, v_2, ..., v_k$ with the property that $\{v_i, v_{i+1}\} \in E$ for every $i \leq k$–1. A walk of endpoints v_1 and v_k is called a v_1–v_k walk or, also, a walk between v_1 and v_k.*

A **walk** is a **trail** if all the edges are different. The following types of trail can be highlighted:

- A **path**, if vertices are not repeated.
- A **circuit**, if it is closed.
- A **cycle** is a circuit (closed) that, by deleting the first vertex, is also a walk (does not repeat vertices). The graphs that do not contain cycles are called **acyclic**.

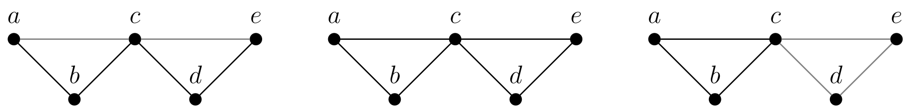

Figure 2.2: Example of walk types in a graph: path, circuit and cycle.

In Figure 2.2 we show an example of the three types of walk descrived for the same graph. In the first graph we show path a, b, c, d, e highlighted. In the middle, we can see that all the edges of the graph form the circuit a, b, c, d, e, c, a. Finally, we have remarked the cycle a, b, c, a in the right graph.

Definition 5. *We say that graphs G and H are* ***isomorphic****, $G \cong H$, if they are are structurally equivalent. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. G and H are* ***identical*** *if and only if $V_G = V_H$ and $E_G = E_H$. G and H are* ***isomorphic****, $G \cong H$, if and only if there exists a bijection $\varphi : V_G \to V_H$ that preserves the adjacencies and the non-adjacencies, that is, $u \sim v \Leftrightarrow \varphi(u) \Leftrightarrow \varphi(v)$. In this case, it is said that φ is a graph* ***isomorphism****.*

Definition 6. *A graph is* ***connected*** *when there is a path between every pair of vertices. This means that in a connected graph, there are no unreachable vertices. A graph G is said to be* ***nonconnected*** *if there exist two nodes in G such that no path in G contains those two nodes at the same time. In a nonconnected graph, each group of connected vertices is said to be a* ***component****. An example of connection is shown in Figure 2.3.*

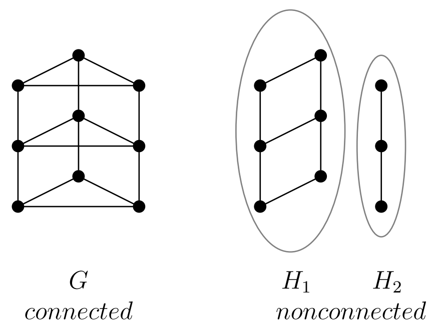

Figure 2.3: Example of a connected graph and a nonconnected graph with its components.

For tree graphs, which are defined in next Subsction 2.1.2, we introduce the following notation:

- **Leaf**: vertex of degree 1.
- **Support vertex**: vertex adjacent to a leaf.
- **Strong support vertex**: vertex adjacent to more than one leaf.

2.1.2 Main families of graphs

- **Empty graphs**. An **empty graph** N_n of order $n \geq 1$ is the graph of n vertices and 0 edges; so that $N_n = (V, \emptyset)$. The graph N_1 is called a **trivial graph**.

- **Complete Graphs**. A **complete** graph K_n is a graph of order n with all possible edges.

- **Bipartite graphs**. A non-empty graph $G = (V, E)$ is **bipartite** if $V = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, so that the existing edges only connect vertices in V_1 with vertices in V_2.

- **Complete bipartite graphs**. The **complete bipartite** graph, denoted by $K_{r,s} = (V_1 \cup V_2, E)$, is a bipartite graph where $|V_1| = r$, $|V_2| = s$, with all the possible edges connecting vertices in V_1 with vertices in V_2.

$$E = \bigcup_{u \in V_1, v \in V_2} \{u, v\}$$

- **Star graphs**. A **star** graph of order $n \geq 3$ is the complete bipartite graph denoted by $K_{1,n-1}$. In star graphs, all nodes are adjacent to the same vertex. $K_{1,n-1} = (V, E)$ such that $V = \{v_1, ..., v_n\}$ and
$$E = \{v_1v_2, v_1v_3, ..., v_1v_n\}.$$

- **Tree grpahs**. A **tree** is a connected graph without any cycle.

  If we delete the condition of connectivity, we obtain a **forest**, that is, a graph composed by a set of trees.

- **Cycle graph**. A **cycle** graph of order $n \geq 3$ is defined as $C_n = (V, E)$, where $V = \{v_1, ..., v_n\}$ and
$$E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}.$$

- **Path graph**. A **path** graph of order $n \geq 2$ denoted as $P_n = (V, E)$ is a tree that can be obtained by the elimination of an edge of the cycle graph C_n. P_n is defined by $V = \{v_1, ..., v_n\}$ and
$$E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}.$$

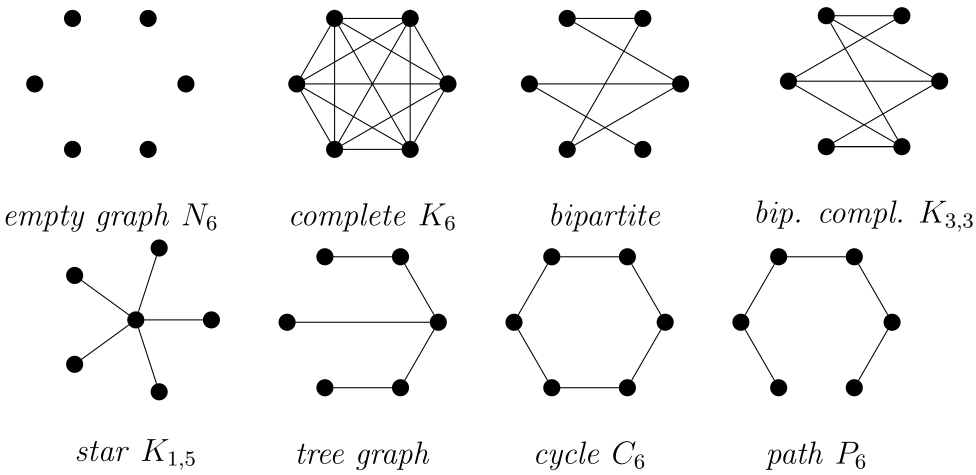

Figure 2.4: Example of some of the main families alf graphs: empty graph, complete graph, bipartite graph, complete bipartite graph, star graph, tree graph, cycle graph and pah graph.

In Figure 2.4 we show an example of each of the families of graphs descrived above. Notice that all the graphs shown have order $N = 6$.

Throughout the work, we will use the notation K_n, $K_{1,n-1}$, C_n, N_n and P_n for complete graphs, star graphs, cycle graphs, empty graphs and path graphs of order n, respectively. We use the notation $u \sim v$ if u and v are adjacent vertices, and $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G, $N(v)$ will denote the set of neighbours or **open neighborhood** of v in G: $N(v) = \{u \in V(G) : \ u \sim v\}$. The **closed neighborhood**, denoted by $N[v]$, equals $N(v) \cup \{v\}$. The subgraph of G induced by a set S of vertices is denoted by $\langle S \rangle$.

For the remainder of the work, definitions will be introduced whenever a concept is needed.

2.2 Elementary operations of graphs

Given a non-trivial graph $G = (V, E)$ diverse operations can be done:

Remove a vertex $u \in V$. In this way we get the graph $G_1 = G–u$, which is the graph $G_1 = (V_1, E_1)$, where $V_1 = V \setminus \{u\}$ and E_1 is the set of edges of G non-incident with u. This operation can be generalised to a set $W \subset V$. That is,

$$G_1 = G–W = (V \setminus W, \{\{a, b\} \in E : a, b \notin W\}).$$

Remove an edge $a \in E$. This is how to get a graph, with the same vertices, defined by $G_2 = G–a = (V, E \setminus \{a\})$; the operation can be trivially generalised to a subset of edges $B \subset E$, in which case $G–B = (V, E \setminus B)$. Given a graph G and an edge $e \in E(G)$, the graph obtained from G by removing the edge e can be denoted by $G - e$, *i.e.*, $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

Add an edge $\{u, v\}$, with u and v being two non-adjacent vertices. In this way we get the graph $G_3 = (V, E \cup \{\{u, v\}\})$. This new graph can be represented by $G + uv$. The process can be generalised to a set B of more than one edge so that $+B = (V, E \cup B)$.

For adding an edge, the condition of non-adjacency of the vertices is fundamental, since the contrary would create a multiple edge and, therefore, would not be in the domain of simple graphs, as they have been defined.

Definition 7. *Given a graph $G = (V, E)$, a graph $G_1 = (V_1, E_1)$ is a* ***subgraph*** *of G if $V_1 \in V$ and $E_1 \in E$. A* ***spanning subgraph*** *of G is subgraph of G containing every vertex of G and a subset of the edges of G: $V_1 = V$ and $E_1 \in E$.*

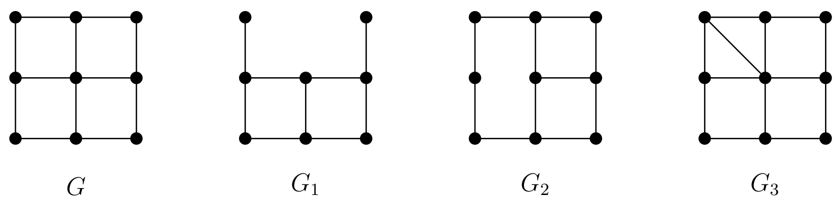

Figure 2.5: Example of vertex removal, edge removal and edge addition.

See the operations described graphically in Figure 2.5. G_1 is the result of removing a vertex from G, G_2 is the removal of an edge of G and G_3 is the addition of an edge to G. Notice that, G_1 is a subgraph of G, G_2 is a spanning subgraph of G and that G is a panning subgraph of G_3, hence G_1 and G_2 are also subgraphs of G_3.

Contract an edge $a = \{u, v\}$. In this case, an edge is replaced by a new vertex. The edge a and the two endpoint vertices u and v of a are removed and are identified in a single new vertex w. This vertex inherits exclusively the adjacencies of vertices u, v.

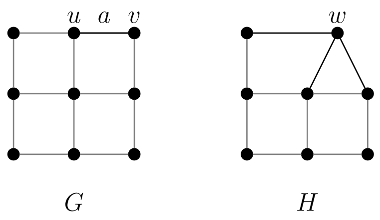

Figure 2.6: Example of edge contraction.

See an example of an edge contraction in Figure 2.6. Notice that in this case the graph H obtained is not a subgraph of the original graph G as in this operation a new vertex is created and the adjacencies of the neighbour vertices are changed.

Definition 8. ***Union of graphs***. *The union of two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$, *where* $V_1 \cap V_2 = \emptyset$ *is a graph*

$$G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2).$$

Remark that in the definition of union of two graphs G_1 and G_2, the interception of the vertex sets of G_1 and G_2 has to be empty; hence the resulting graph $G = G_1 \cup G_2$ is a non-connected graph.

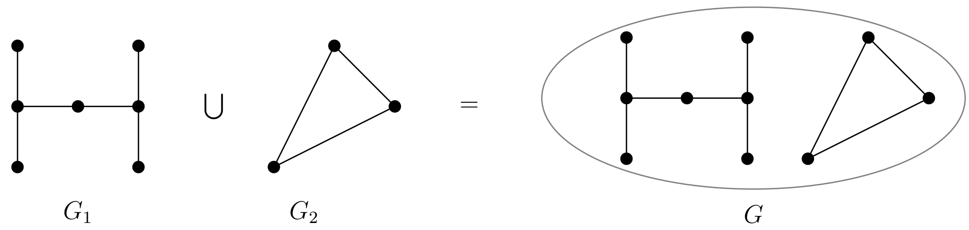

Figure 2.7: Example of union of two graphs.

See in Figure 2.7 a representation of the union operation of two graphs. Notice that if G_1 and G_2 are connected graphs, G will be a non-connected graph of components G_1 and G_2.

Definition 9. ***Join of graphs***. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two graphs such that* $V_1 \cap V_2 = \emptyset$. *The join* $G_1 + G_2$ *is the graph that has the vertices and the edges of the original graphs, in addition to the edges that connect all the vertices of* G_1 *with all the vertices of* G_2:

$$G = G_1 + G_2 = (V_1 \cup V_2, (E_1 \cup E_2 \cup \{\{u, v\} : u \in V_1, v \in V_2\})).$$

Remark that, as in the case of the union opertion, in the definition of join of two graphs G_1 and G_2, the interception of the vertex sets of G_1 and G_2 has to be empty. Notice, that the resulting graph is always connected independently of the connection properties of the factor graphs.

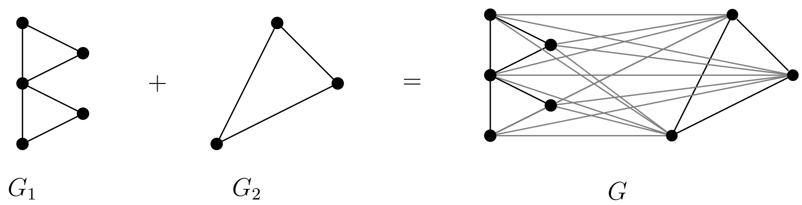

Figure 2.8: Example of join of two graphs.

Join operation is represented in Figure 2.8 where the edges generated by the operation are remarked in grey.

Definition 10. ***Product of graphs***. *Graphs are composed by two sets of different elements: set of vertices V and set of edges E. In mathematics, a **graph product** is a binary **spread operation** on those sets. This operation can be defined in multiple ways according to the nature of the product; each strategy defines how $V(GH)$ and $E(GH)$ are built according to the properties needed.*

From now on we will refer to the vertices of GH according to the original vertices of G and H. Let u be a vertex $u \in V(G)$ and v a vertex $v \in V(H)$. We identify $(u, v) \in V(GH)$ as the vertex corresponding to the product of u and v.

2.2.1 Cartesian product

Given two graphs $G = (V_G, E_G)$, $H = (V_H, E_H)$, the Cartesian product $G \square H = (V_G \times V_H, E)$ is defined so that two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if they satisfy any of the following conditions:

(i) $u_1 = u_2$ and $v_1 \sim v_2$, or

(ii) $u_1 \sim u_2$ and $v_1 = v_2$

To imagine or graphically represent Cartesian product $G \square H$ we can think that by each vertex of H we put a copy of graph G and each vertex of the i-th copy of G will be adjacent to its twin in j-th copy of G if and only if i and j are adjacent in H. See this strategy illustrated in Figure 2.9.

For more information on structure and properties of the Cartesian product of graphs we refer the reader to [25, 30].

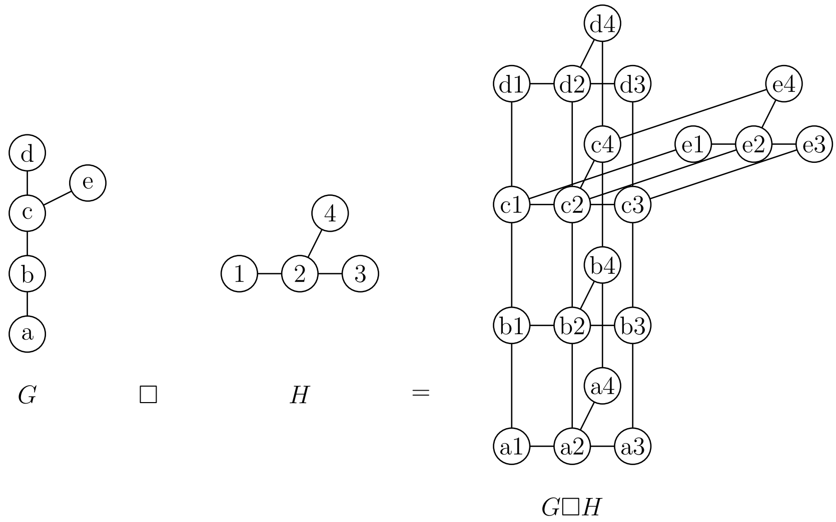

Figure 2.9: Cartesian product of graphs

2.2.2 Lexicographic product

Let G and H be two graphs. The *lexicographic product* of G and H is the graph $G \circ H$ whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and $(u, v)(x, y) \in E(G \circ H)$ if and only if $ux \in E(G)$ or $u = x$ and $vy \in E(H)$.

A way to represent graphically lexicographic product $G \circ H$ we can think that by each vertex of G we put a copy of graph H. All copies of H will contain the same adjacencies as H and these copies will be joined if the vertices of G corresponding to that copies are adjacent.

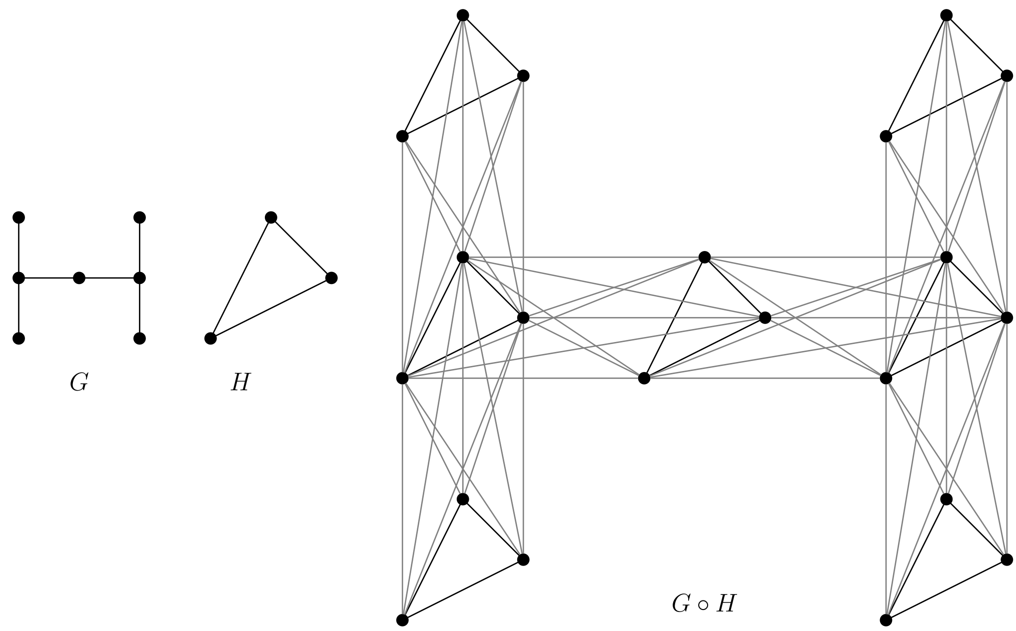

Figure 2.10: Example of lexicographic product of two graphs.

Notice that for any $u \in V(G)$ the subgraph of $G \circ H$ induced by $\{u\} \times V(H)$ is isomorphic to H. For simplicity, we will denote this subgraph by H_u, and if a vertex of G is denoted by u_i, then the referred subgraph will be denoted by H_i. See this construction in Figure 2.10 where all copies H_u of H are remarked in black for each $u \in V(G)$.

For basic properties of the lexicographic product of two graphs we suggest the books [25, 30].

2.3 Protection of graphs

The following approach to protection of a graph was described by Cockayne et Al. [19]. Suppose that one or more entities are stationed at some of the vertices of a graph G and that an entity at a vertex can deal with a problem at any vertex in its closed neighborhood. In general, an entity could consist of an observer, a robot, a guard, a legion, and so on. Informally, we say that G is protected under a given placement of entities if there exists at least one entity available to handle a problem at any vertex.

Consider a function $f \to V\{0, 1, 2, \dots\}$ where $f(v)$ is the number of entities at v, and for $i \in \{0, 1, 2, \dots\}$ $V = \{v \in V | f(v) = i\}$. We will identify f with the partition V induced by f and write $f = (V_0, V_1, V_2, \dots)$. The **weight** of f, $w(f) = \sum_{v \in V} f(v) = \sum_{i \geq 1} i|V_i|$ is the total number of entities used by f. A vertex $v \in V(G)$ is *unprotected* with respect to f if $f(v) = 0$ and $f(u) = 0$ for every vertex u adjacent to v. We say that G is *protected* under the function f if f has no unprotected vertices, *i.e.*, G is protected if there is at least one entity available to handle the problem at any vertex. Formally, $f = (V_0, V_1, V_2, \dots)$ is *safe* if each $v \in V_0$ is adjacent to at least one vertex of $V - V_0$. The set of vertices containing entities, $V - V_0$, is said to be a dominating set of G.

Cockayne et Al. [18] proposed four properties of such functions under which the entire graph may be protected according to a certain strategy. In each case the parameter of interest will be the minimum weight of (*i.e.* the minimum number of entities used by) a function in the subclass. There can be more domination definitions but we will focus on the ones used for the aim of this work. We may refer to other strategies during the study, in such case, the definition will be introduced when necessary.

1. **Domination**

   We say that $f(V_0, V_1)$ is a *Dominating Function (DF)* if G is protected under f. Obviously, $f(V_0, V_1)$ is a DF if and only if V_1 is a dominating set. The *domination number*, denoted by $\gamma(G)$ is the minimum cardinality among all dominating sets of G. This method of protection has been studied extensively[26, 27].

2. **Roman domination**

   A *Roman Dominating Function (RDF)* is a function $f(V_0, V_1, V_2)$ such that for every $v \in V_0$ there exists a vertex $u \in V_2$ which is adjacent to v. The *Roman domination number*, denoted by $\gamma_R(G)$, is the minimum weight among all Roman dominating functions on G. This concept of protection has historical motivation [36] and was formally proposed by Cockayne et Al. in [20].

3. **weak Roman domination**

   A *Weak Roman Dominating Function (WRDF)* is a function $f(V_0, V_1, V_2)$ such that for every v with $f(v) = 0$ there exists a vertex u adjacent to v such that $f(u) \in \{1, 2\}$ and the function $f' : V(G) \longrightarrow \{0, 1, 2\}$ defined by $f'(v) = 1$, $f'(u) = f(u) - 1$ and $f'(z) = f(z)$ for every $z \in V(G) \setminus \{u, v\}$, has no unprotected vertices. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight among all weak Roman dominating functions on G. A WRDF of weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. This concept of protection was introduced by Henning and Hedetniemi [28] and studied further in [17, 18].

4. **Secure Roman domination**

   A *Secure Dominating Function (SDF)* is a WRDF $f(V_0, V_1, V_2)$ in which $V_2 = \emptyset$. In this case, it is convenient to define this concept of save graph by the properties of V_1. Obviously $f(V_0, V_1)$ is a secure dominating function if and only if V_1 is a dominating set and for every $v \in V_0$ there exists $u \in V_1$ which is adjacent to v and $(V_1 \setminus \{u\}) \cup \{v\}$ is a dominating set. In such a case, V_1 is said to be a *secure dominating set*. The *secure domination number*, denoted by $\gamma_s(G)$, is the minimum cardinality among all secure dominating sets. A secure dominating function of weight $\gamma_s(G)$ is called a $\gamma_s(G)$-function. Analogously, a secure dominating set of cardinality $\gamma_s(G)$ is called a $\gamma_s(G)$-set. This concept of protection was introduced by Cockayne et al. in [19], and studied further in [17, 18].

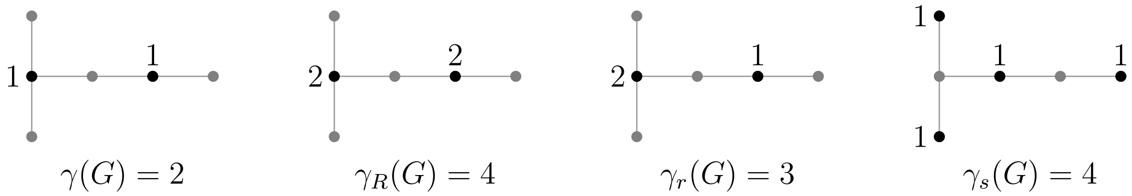

Figure 2.11: Placements of entities corresponding to the four subclasses of dominating functions.

Figure 2.11 shows the behaviour of protection according to each of the four subclasses of domination for the same tree. Notice that $2 = \gamma(G) < \gamma_r(G) < \gamma_R(G) = 4$ and that $2 = \gamma(G) < \gamma_r(G) < \gamma_s(G) = 4$.

Chapter 3

General bounds

In this Chapter we discuss some basic remarks on the protection of graphs. We focus on the similar results for weak Roman domination number γ_r and secure domination number γ_s of a graph.

For nonconnected graphs we have the following remark.

Remark 1. *For any graph G of k components, $G_1, G_2, \ldots, G_k$,*

(i) $\gamma_r(G) = \sum_{i=1}^{k} \gamma_r(G_i)$,

(ii) $\gamma_s(G) = \sum_{i=1}^{k} \gamma_s(G_i)$.

According to the remark above, we can restrict ourselves to the case of connected graphs.

Proposition 2. *We would like to emphatize that the following inequalty chains hold for any graph G:*

(i) $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

(ii) $\gamma(G) \leq \gamma_r(G) \leq \gamma_s(G)$.

The problem of characterizing the graphs with $\gamma_r(G) = \gamma(G)$ was solved by Henning and Hedetniemi [28]. The inequality chain *(ii)* has motivated us to obtain the following result, which shows that the problem of characterizing the graphs with $\gamma_s(G) = \gamma(G)$ is already solved.

Theorem 3. *Let G be a graph. The following statements are equivalent.*

(i) $\gamma_r(G) = \gamma(G)$.

(ii) $\gamma_s(G) = \gamma(G)$.

Proof. By Proposition 2 *(ii)*, $\gamma_s(G) = \gamma(G)$ leads to $\gamma_r(G) = \gamma(G)$. Now, if $\gamma_r(G) = (G)$, then for any $\gamma_r(G)$-function $f(V_0, V_1, V_2)$ we have $V_2 = \emptyset$;, as $V_1 \cup V_2$ is a dominating set and $\gamma(G) = \gamma_r(G) = |V_1| + 2|V_2| \geq |V_1| + |V_2| \geq \gamma(G)$. Hence, V_1 is a secure dominating set, which implies that $\gamma(G) = |V_1| \geq \gamma_s(G) \geq \gamma(G)$. Therefore, $\gamma_s(G) = \gamma(G)$. □

As observed in [28] any $\gamma_r(G-e)$-function is a WRDF for G. Similarly, any $\gamma_s(G-e)$-set is a secure dominating set for G. Therefore, the following basic result follows.

Remark 4. *For any spanning subgraph H of a graph G,*

(i) *[28]* $\gamma_r(G) \leq \gamma_r(H)$.

(ii) $\gamma_s(G) \leq \gamma_s(H)$.

Proposition 5. *For any $n \geq 4$,*

(i) *[28]* $\gamma_r(C_n) = \gamma_r(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.

(ii) *[19]* $\gamma_s(C_n) = \gamma_s(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.

Definition 11. *A Hamiltonian graph is a graph such that contains a cycle that passes through each node exactly once. An example of a Hamiltonian graph is shown in Figure 3.1.*

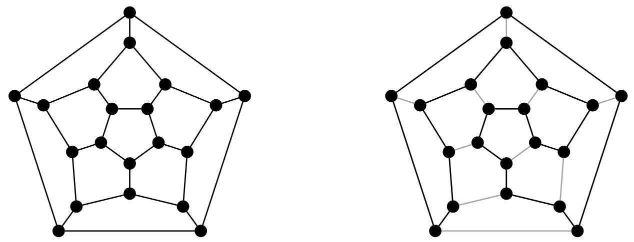

Figure 3.1: Example of Hamiltonian graph and its Hamiltonian cycle.

By Remark 4 and Proposition 5 we deduce the following result.

Theorem 6. *For any Hamiltonian graph G of order $n \geq 4$,*

$$\gamma_r(G) \leq \gamma_s(G) \leq \left\lceil \frac{3n}{7} \right\rceil.$$

Obviously, the bound above is tight, as it is achieved for $G \cong C_n$ having order $n \geq 4$.

Chapter 4

On weak Roman domination

This Chapter explores particular results for weak Roman domination and corresponds to the work presented in final degree thesis.

To start, we specify the graphs having domination number lower than three and define the so-called weak Roman graphs. In next Subsection 4.1, we study weak Roman domination in lexicographic product graphs compare it with other known metrics.

Remark 7. *Let G be a graph of order n. Then $\gamma_r(G) = 1$ if and only if $G \cong K_n$.*

According to this remark, for any noncomplete graph G we have that $\gamma_r(G) \geq 2$. The limit case of this trivial bound will be discussed in the following remark. From now on we say that a set $\{a, b\} \subseteq V(G)$ satisfies Property $\mathcal{P}$ if the following conditions hold.

- $\{a, b\}$ is a dominating set.
- If $x \in V(G) \setminus N[a]$, then $\{x, a\}$ is a dominating set.
- If $x \in V(G) \setminus N[b]$, then $\{x, b\}$ is a dominating set.
- If $x \in N(a) \cap N(b)$, then $\{x, a\}$ is a dominating set or $\{x, b\}$ is a dominating set.

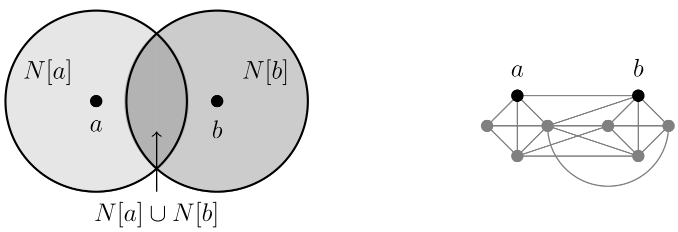

Figure 4.1: On the left side, a diagram of the vertices of a graph with a set accomplishing property $\mathcal{P}$. On the right, an example graph.

This property $\mathcal{P}$ is illustrated diagramatically together with an example in Figure 4.1. Notice that in the graphs accomplishing property, all vertices in the graph form a clique with a or b as shown in the figure above.

Remark 8. *A graph G of order n satisfies $\gamma_r(G) = 2$ if and only if $G \ncong K_n$ and at least one of the following conditions holds.*

(i) $\gamma(G) = 1$.

(ii) There exists $\{a, b\} \subseteq V(G)$ which satisfies Property $\mathcal{P}$.

Proof. (Sufficiency) Assume that $\gamma_r(G) = 2$ and let $f(X_0, X_1, X_2)$ be a $\gamma_r(G)$-function. By Remark 7, $G \ncong K_n$. Notice that $|X_1| + 2|X_2| = 2$. Thus, if $|X_2| = 1$, then $\gamma(G) = 1$. Now, if $X_1 = \{a, b\}$, then $\{a, b\}$ is a dominating set and for every $x \in V(G) \setminus \{a, b\}$, the movement of a legion from a to x, or from b to x, does not produce unprotected vertices. This implies that if $x \in V(G) \setminus N[b]$, then $\{x, b\}$ is a dominating set, if $x \in V(G) \setminus N[a]$, then $\{x, a\}$ is a dominating set, and if $x \in N(a) \cap N(b)$, then $\{x, a\}$ is a dominating set or $\{x, b\}$ is a dominating set. Hence, $\{a, b\}$ satisfies Property $\mathcal{P}$.

(Necessity) Let $G \ncong K_n$. We first assume that $\{a\}$ is a dominating set of G. In this case we can define a WRDF $f(W_0, W_1, W_2)$ by $W_0 = V(G) \setminus \{a\}$, $W_1 = \emptyset$ and $W_2 = \{a\}$, as the movement of a legion from a to any vertex $x \in V(G) \setminus \{a\}$ does not produce unprotected vertices. Finally, assume that $\{a, b\}$ satisfies Property $\mathcal{P}$. In this case we can define a WRDF $f(W_0, W_1, W_2)$ by $W_0 = V(G) \setminus \{a, b\}$, $W_1 = \{a, b\}$ and $W_2 = \emptyset$. Obviously, by definition of property $\mathcal{P}$, the movement of a legion from a (or from b) to $x \in W_0$, does not produce unprotected vertices. In both cases we can conclude that $\gamma_r(G) \leq |X_1| + 2|X_2| = 2$ and the equality holds by Remark 7. □

Graphs with $\gamma_R(G) = 2\gamma(G)$ are called Roman graphs [28]. We say that G is a weak Roman graph if $\gamma_r(G) = 2\gamma(G)$. Notice that any weak Roman graph is a Roman graph. In general, the converse does not hold. For instance, the graph shown in Figure 2.11 is a Roman graph, as $\gamma_R(G) = 2\gamma(G) = 4$, while it is not a weak Roman graph as $\gamma_r(G) = 3$.

Lemma 9. *[28] If T is a tree with a unique $\gamma(T)$-set S, and if every vertex in S is a strong support vertex, then T is a weak Roman tree.*

The reader is refereed to [28] for a complete characterization of all weak Roman forest.

4.1 Lexicographic product graphs

This subsection covers the results on weak Roman domination in lexicographic product graphs. We divide it in three subparts according to the nature of the result. First, we have an overview on upper bounds of the domination in Section 4.1.1 followed by the lower bounds in Section 4.1.2. Finally we show in Section 4.1.3 closed formulae for the product. In order to facilitate the lecture of the work, demonstration of Theorem 37 is moved to Annex 1.

For any $u \in V(G)$ and any WRDF f on $G \circ H$ we define

$$f(H_u) = \sum_{v \in V(H)} f(u, v) \text{ and } f[H_u] = \sum_{x \in N[u]} f(H_x).$$

Remark 10. *Let G and H be two graphs. The following assertions hold.*

- *$G \circ H$ is connected if and only if G is connected.*
- *If $G = G_1 \cup \ldots \cup G_t$, then $G \circ H = (G_1 \circ H) \cup \ldots \cup (G_t \circ H)$.*

From Remarks 1 and 10 we deduce the following result.

Remark 11. *For any graph G of component $G_1, G_2, \ldots, G_k$ and any graph H,*

$$\gamma_r(G \circ H) = \sum_{i=1}^{k} \gamma_r(G_k \circ H).$$

The following result is a direct consequence of Remark 4.

Remark 12. *Let G be a connected graph of order n and let H be a nonempty. For any spanning subgraph G_1 of G,*

$$\gamma_r(K_n \circ H) \leq \gamma_r(G \circ H) \leq \gamma_r(G_1 \circ H).$$

In particular, if G is a Hamiltonian graph, then

$$\gamma_r(G \circ H) \leq \gamma_r(C_n \circ H).$$

4.1.1 Upper bounds on $\gamma_r(G \circ H)$

Definition 12. *A total dominating set of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. For each vertex $u \in V(G)$ there exists $s \neq u$ such that $s \in S$ and u and s are neighbours, even if $u \in S$. The total domination number of G, denoted by $\gamma_t(G)$, is the cardinality of a smallest total dominating set, and we refer to such a set as a $\gamma_t(G)$-set.*

Notice that for any graph G with no isolated vertex,

$$\gamma_r(G) \leq 2\gamma(G) \leq 2\gamma_t(G). \tag{4.1}$$

The reader is referred to the book [29] for details on total domination in graphs where fundamentals of this type of domination are provided and explored. Two examples of this domination are shown in the graphs of Figure 4.2 where the vertices of the dominating set are remarked.

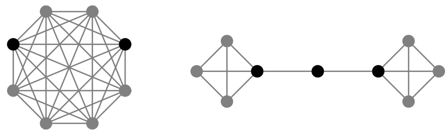

Figure 4.2: Example of total domination in different graphs.

Notice that if D is a total dominating set of G and $h \in V(H)$, then $D \times \{h\}$ is a dominating set of $G \circ H$, so that $\gamma(G \circ H) \leq \gamma_t(G)$. Hence, from the first inequality in chain (4.1) we deduce the following theorem, which can also be derived from the inequity $\gamma_R(G \circ H) \leq 2\gamma_t(G)$ observed in [37] and the second inequality in Remark 2 *(i)*.

Theorem 13. *If G is a graph with no isolated vertex, then for any graph H,*

$$\gamma_r(G \circ H) \leq 2\gamma_t(G).$$

The total domination number of a Path P_n is known and is easy to compute. For every integer $n \geq 3$ we have $\gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. We will show in Corollary 39 that if $\gamma(H) \geq 4$, then $\gamma_r(P_n \circ H) = 2\gamma_t(P_n)$. Thus, the bound above is tight. Furthermore, as we will show in Proposition 34, if $n \geq 3$ and $\gamma(H) \geq 4$, then $\gamma_r(K_{1,n-1} \circ H) = 4 = 2\gamma_t(K_{1,n-1})$.

Notice that $K_{1,n-1}$ is a graph of diameter two and minimum degree $\delta = 1$. In general, for any graph G of diameter two and minimum degree δ, the total domination number of G is bounded above by $\delta + 1$. Moreover, if G is a graph of order n with no isolated vertex and maximum degree $\Delta \geq n - 2$, then $\gamma_t(G) = 2$. Therefore, the following result is a direct consequence of Theorem 13.

Corollary 14. *The following assertions hold for any graph H.*

- *If G is a graph of order n with no isolated vertex and maximum degree $\Delta \geq n - 2$, then $\gamma_r(G \circ H) \leq 4$.*
- *If G has diameter two and minimum degree δ, then $\gamma_r(G \circ H) \leq 2(\delta + 1)$.*

It was shown in [12] that for any connected graph of order $n \geq 3$, $\gamma_t(G) \leq \frac{2}{3}n$. Hence, Theorem 13 leads to the following result.

Corollary 15. *For any connected graph G of order $n \geq 3$ and any graph H,*

$$\gamma_r(G \circ H) \leq 2 \left\lfloor \frac{2n}{3} \right\rfloor .$$

We will show in Proposition 36 that the bound above is tight.

Chellali and Haynes [11] established that the total domination number of a tree T of order $n \geq 3$ is bounded above by $(n+s)/2$, where s is the number of support vertex of T. Therefore, Theorem 13 leads to the following corollary.

Corollary 16. *For any graph H and any tree T of order $n \geq 3$ having s support vertex,*

$$\gamma_r(T \circ H) \leq n + s.$$

The bound above is tight. For instance, Proposition 36 shows that for any $n = 3k$ and any graph H with $\gamma(H) \geq 4$, $\gamma_r(T_n \circ H) = n + s = 4k$, where T_n is a comb graph. *This family of graphs is defined in Definition 19 for the aim of this work.*

As stated by Goddard and Henning [23], if G is a planar graph with diameter two, then $\gamma_t(G) \leq 3$. Hence, as an immediate consequence of Theorem 13, we have the following result.

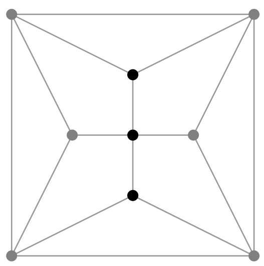

Figure 4.3: A planar graph of diameter two.

Corollary 17. *If G is a planar graph of diameter two, then for any graph H,*

$$\gamma_r(G \circ H) \leq 6.$$

The bound above is achieved, for instance, for the planar graph G shown in Figure 4.3 and any graph H with $\gamma(H) \geq 4$. An optimum placement of legions in $G \circ H$ can be done by assigning two legions to the copies of H corresponding to the black-coloured vertices of G.

Corollary 18. *For any graph G with no isolated vertex and any noncomplete graph H,*

$$\gamma_r(G \circ H) \le 4\gamma(G).$$

Proof. It is well known that for every graph G with no isolated vertex,$\gamma_t(G) \le 2\gamma(G)$ (see, for instance, [10]). Hence, by Theorem 13 we have $\gamma_r(G \circ H) \le 4\gamma(G)$. Therefore, the result follows. □

The bound $\gamma_r(G \circ H) \le 4\gamma(G)$ is achieved, for instance, for graphs G and H satisfying the assumptions of Theorem 30

Theorem 19. *For any graph G and any noncomplete graph H,*

$$\gamma_r(G \circ H) \le \gamma(G)\gamma_r(H).$$

Proof. Let $f_1(V_0, V_1, V_2)$ be a $\gamma_r(H)$-function and X a $\gamma(G)$-set. Notice that $\gamma_r(H) \ge 2$, as H is not complete. It is readily seen that $f(W_0, W_1, W_2)$ defined by $W_1 = X \times V_1$ and $W_2 = X \times V_2$ is a WRDF of $G \circ H$. Hence,

$$\gamma_r(G \circ H) \le |X \times V_1| + 2|X \times V_2| = |X|(|V_1| + 2|V_2|) = \gamma(G)\gamma_r(H).$$

Therefore, the result follows. □

The bound $\gamma_r(G \circ H) \le \gamma(G)\gamma_r(H)$ is achieved, for instance, for any comb graph T_{3k} defined in Definition 19 and any graph H with $\gamma_r(H) = 4$. Besides, the bound is attained for any G and H satisfying the assumptions of Theorem 28.

Definition 13. *A double total dominating set of a graph G with minimum degree greater than or equal to two is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least two vertices in S, [29]. The double total domination number of G, denoted by $\gamma_{2t}(G)$, is the cardinality of a smallest double total dominating set, and we refer to such a set as a $\gamma_{2t}(G)$-set. Two examples of double total dominating set are shown in Figure 4.4.*

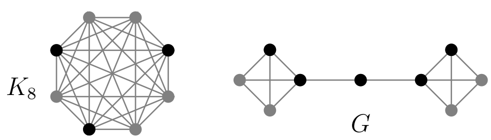

Figure 4.4: Example of double total domination in different graphs.

Notice that the graphs used to describe double total domination are the same as the ones used to define the total domination, where the corresponding domination numbers are different. For the graph K_8 we have that $2 = \gamma_t(K_8) \leq \gamma_{2t}(K_8) = 3$, and for the right hand side graph we have that $3 = \gamma_t(G) \leq \gamma_{2t}(G) = 5$.

Theorem 20. *Let G be a graph of minimum degree greater than or equal to two. The following assertions hold.*

(i) $\gamma_r(G) \leq \gamma_{2t}(G)$.

(ii) For any graph H, $\gamma_{2,t}(G \circ H) \leq \gamma_{2t}(G)$.

(iii) For any graph H, $\gamma_r(G \circ H) \leq \gamma_{2t}(G)$.

Proof. For every $\gamma_{2t}(G)$-set. S we can define a WRDF $f(X_0, X_1, X_2)$ on G by $X_0 = V(G) \setminus S$, $X_1 = S$ and $X_2 = \emptyset$. Hence, *(i)* follows.

Now, let D be a $\gamma_{2t}(G)$-set and $y \in V(H)$. Thus, for every $(x, y) \in V(G) \times V(H)$, there exist $a, b \in D \cap N(x)$, which implies that $(a, y), (b, y) \in (D \times \{y\}) \cap N(x, y)$, and so $D \times \{y\}$ is a double total dominating set of $G \circ H$. Hence, *(ii)* follows.

Finally, from *(i)* and *(ii)* we deduce *(iii)*, as $\gamma_r(G \circ H) \leq \gamma_{2,t}(G \circ H) \leq \gamma_{2t}(G)$. □

In order to show an example of graphs where $\gamma_r(G \circ H) = \gamma_{2t}$, we define the family $\mathcal{G}$ as follows. A graph $G_{r,s} = (V, E)$ belongs to $\mathcal{G}$ if and only if there exit two positive integers r, s such that $V = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s\}$ and $E = \{x_1 y_i : 1 \leq i \leq r\} \cup \{x_1 z_i : 1 \leq i \leq s\} \cup \{x_2 y_i : 1 \leq i \leq r\} \cup \{x_3 z_i : 1 \leq i \leq s\} \cup \{x_2 x_3\}$. Figure 4.5 shows the graph $G_{4,4}$.

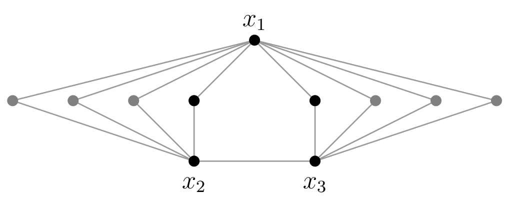

Figure 4.5: The set of black-coloured vertices is a double total dominating set of $G_{4,4}$.

It is not difficult to check that for any graph $G_{r,s} \in$ family $\mathcal{G}$ and any graph H with $\gamma(H) \geq 3$ we have $\gamma_r(G_{r,s} \circ H) = 5 = \gamma_{2,t}(G_{r,s})$.

Corollary 21. *For any graph H and any graph G of order n and minimum degree greater than or equal to two,*

$$\gamma_r(G \circ H) \leq n.$$

As we will show in Corollary 38, the bound above is tight.

4.1.2 Lower bounds on $\gamma_r(G \circ H)$

In order to deduce or next result we need to state the following basic lemma.

Lemma 22. *Let G be a graph and H a noncomplete graph. For any $u \in V(G)$ and any $\gamma_r(G \circ H)$-function f,*

$$f[H_u] = \sum_{x \in N[u]} f(H_x) \geq 2.$$

Proof. Suppose that f is a $\gamma_r(G \circ H)$-function and there exists $u \in V(G)$ such that $f[H_u] \leq 1$. If $f[H_u] = 1$, then the placement of a legion in a non-universal vertex of H_u produces unprotected vertices, which is a contradiction. Now, if $f[H_u] = 0$, then there are unprotected vertices in H_u, which is a contradiction again. Therefore, the result follows. □

Definition 14. *A set $X \subseteq V(G)$ is called a 2-packing set if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$. The 2-packing number $\rho(G)$ is the cardinality of any largest 2-packing set of G. A 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-set. An example of a $\rho(G)$-set is shown in Figure 4.6, where $\rho(P_3 \odot N_3) = 3$.*

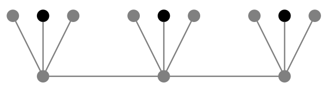

Figure 4.6: The set of black-coloured vertices are the ones forming the 2-packing set.

Notice that, In general, for any graph G of order n and any graph H, $\rho(G \odot H) = n$.

Theorem 23. *For any graph G of minimum degree $\delta \geq 1$ and any noncomplete graph H,*

$$\gamma_r(G \circ H) \geq \max\{\gamma_r(G), \gamma_t(G), 2\rho(G)\}.$$

Proof. Let $f(W_0, W_1, W_2)$ be a $\gamma_r(G \circ H)$-function. In order to show that $\gamma_r(G \circ H) \geq \gamma_r(G)$, we will show that there exists a WRDF $f_1(X_0, X_1, X_2)$ of G where $X_0 = \{x : (x, y) \in W_0\}$, $X_2 = \{x : (x, y) \in W_2 \text{ or } |\{x\} \times W_1| \geq 2\}$ and $X_1 = V(G) \setminus (X_0 \cup X_2)$. Notice that, since $W_1 \cup W_2$ is a dominating set of $G \circ H$, $X_1 \cup X_2$ is a domination set of G. Now, for every $(x, y) \in W_0$ there exists $(x', y') \in N(x, y) \cap (W_1 \cup W_2)$ and a function $f' : V(G \circ H) \longrightarrow \{0, 1, 2\}$ defined by $f'(x', y') = f(x', y') - 1$, $f'(x, y) = 1$ and $f'(a, b) = f(a, b)$ for every $(a, b) \notin \{(x, y), (x', y')\}$, which has no unprotected vertex. Hence, the function $f_1' : V(G) \longrightarrow \{0, 1, 2\}$ defined by $f_1'(x') = f_1(x') - 1$, $f_1'(x) = 1$ and $f_1'(a) = f_1(a)$ for every $a \notin \{x, x'\}$ has no unprotected vertex. Thus, $\gamma_r(G \circ H) \geq \gamma_r(G)$.

Now, let $X \subset V(G)$ be a $\rho(G)$-set. By Lemma 22 we have that

$$\gamma_r(G \circ H) = w(f) = \sum_{u \in V(G)} f(H_u) \geq \sum_{u \in X} f[H_u] \geq 2|X| = 2\rho(G).$$

In order to prove that $\gamma_r(G \circ H) \geq \gamma_t(G)$, we define $U_i = \{x \in V(G) : f(H_x) = i\}$, where $i \in \{0,1\}$, and $U_2 = \{x \in V(G) : f(H_x) \geq 2\}$. By Lemma 22 we have that if $x \in U_1$, then there exists $x' \in N(x) \cap (U_1 \cup U_2)$. Now, let $U_2^* = \{x \in U_2 : \sum_{x' \in N(x)} f(H_{x'}) = 0\}$. Since $\delta \geq 1$, there exists $U_0^* \subseteq U_0$ such that $|U_0^*| \leq |U_2^*|$ with the property that for every $x \in U_2^*$ there exists $x^* \in U_0^* \cap N(x)$. Notice that $U_1 \cup U_2 \cup U_0^*$ is a total dominating set. Therefore,

$$\gamma_r(G \circ H) = w(f) = \sum_{u \in V(G)} f(H_u) \geq |U_1 \cup U_2 \cup U_0^*| \geq \gamma_t(G),$$

as required. □

On the one hand, an example of a graph with $\gamma_r(G) > \max\{\gamma_t(G), 2\rho(G)\}$ is the graph shown in Figure 4.7 (on the right), where $\gamma_r(G) = 5$, $\gamma_t(G) = 4$ and $2\rho(G) = 4$. On the other hand, an example of a graph with $2\rho(G) > \max\{\gamma_r(G), \gamma_t(G)\}$ is the path graph P_n, $n \geq 4$, as $\gamma_r(P_n) = \left\lceil \frac{3n}{7} \right\rceil$, $\gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$ and $2\rho(P_n) = 2\gamma(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$. Finally, for the graph shown in Figure 4.7 (on the left) we have $\gamma_t(G) = 5 > 4 = \max\{\gamma_r(G), \rho(G)\}$.

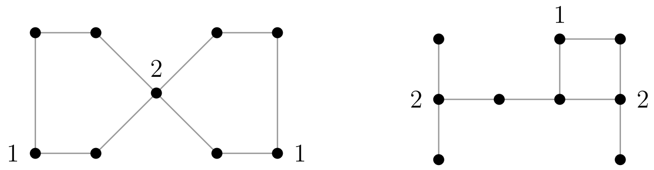

Figure 4.7: $\gamma_r(G) = 4$, the labels correspond to an optimum placement of legions.

We will discuss several cases in which $\gamma_r(G \circ H) = \max\{\gamma_r(G), \gamma_t(G), 2\rho(G))\}$ is achieved in the following subsection 4.1.3.

It is well known that for any graph G, $\gamma(G) \geq \rho(G)$. Meir and Moon [34] showed in 1975 that $\gamma(T) = \rho(T)$ for any tree T. We remark that in general, these $\gamma(T)$-sets and $\rho(T)$-sets are not identical. Notice that for any weak Roman tree T we have $\gamma_r(T) = 2\rho(T)$, while if T is not a weak Roman tree, then $\gamma_r(T) < 2\gamma(G) = 2\rho(T)$.

Corollary 24. *For any tree T and any noncomplete graph H,*

$$\gamma_r(T \circ H) \geq 2\gamma(T).$$

The bound above is achieved for any tree T and any graph H satisfying the assumptions of Theorem 29.

4.1.3 Closed formulae for $\gamma_r(G \circ H)$

In this Section we will have an overview on the colsed formulae and demonstration of tight bounds for lexicographic product graphs. Remind that proof of Theorem 37 is moved to Annex 1 due to its huge extension.

To begin this section we consider the case of lexicographic product graphs in which the second factor is a complete graph.

Proposition 25. *For any graph G and any integer $n \geq 1$,*

$$\gamma_r(G \circ K_n) = \gamma_r(G).$$

Proof. The result is straightforward. We leave the details to the reader. □

From Theorems 13 and 23 we have the following result.

Theorem 26. *For any graph G with $\gamma_t(G) = \frac{1}{2}\max\{\gamma_r(G), 2\rho(G)\}$ and any noncomplete graph H,*

$$\gamma_r(G \circ H) = 2\gamma_t(G).$$

To show some families of graphs for which $\gamma_r(G) = 2\gamma_t(G) = 2\rho(G)$, we introduce the corona product of two graphs.

Definition 15. *Let G be a graph of order n and let H be a graph. The corona product of G and H, denoted by $G \odot H$, was defined in [22] as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i-th copy of H with the i-th vertex of G.*

Theorem 27. *For any graph G with no isolated vertex and any noncomplete graph H,*

$$\gamma_r(G \odot H = 2\gamma_t(G \odot H) = 2\rho(G \odot H).$$

Proof. Since $\gamma(G \odot H) = |V(G)|$, we have that $\gamma_r(G \odot H) \leq 2|V(G)|$. Now, we denote by $\langle g_i \rangle + H$ the subgraph of $G \odot H$ induced by $g_i \in V(G)$ and the vertex set of the i-th copy of H. Since H has two nonadjacent vertices and g_i is the only vertex of $\langle g_i \rangle + H$ which is adjacent to some vertex outside $\langle g_i \rangle + H$, we deduce that every $\gamma_r(G \odot H)$-function assigns at least two legions to the vertex set of $\langle g_i \rangle + H$, which implies that $\gamma_r(G \odot H) \geq 2|V(G)|$. Now, since G is a graph with no isolated vertex, $V(G)$ is a total dominating set. Hence, $\gamma_r(G \odot H) = 2|V(G)| = 2\gamma_t(G \odot H)$.

The proof of the equality $\gamma_t(G \odot H) = \rho(G \odot H)$ is straightforward. □

If $\gamma_r(G) = 2\gamma(G)$, then for the Cocktail-party graph $K_{2k} - F$ we have $\gamma_r(G \circ (K_{2k} - F)) = \gamma_r(G)$. This example is a particular case of the next result which is derived from Theorems 19 and 23.

Theorem 28. *For any weak Roman graph G and any graph H such that $\gamma_r(H) = 2$,*

$$\gamma_r(G \circ H) = 2\gamma(G).$$

The study of weak Roman graphs was initiated in [28] by Henning and Hedetniemi, where they characterized forests for which the equality holds. The general problem of characterizing all weak Roman graphs remains open.

From Lemma 9 and Theorem 28 we derive the following result.

Theorem 29. *If T is a tree with a unique $\gamma(T)$-set S, and if every vertex in S is a strong support vertex, then for any graph H with $\gamma_r(H) = 2$,*

$$\gamma_r(T \circ H) = 2\gamma(T).$$

Our next result shows that the inequality $\gamma_r(G \circ H) \leq 4\gamma(G)$ stated in Corollary 18 is tight.

Theorem 30. *If G is a graph with $\gamma_t(G) = 2\gamma(G)$ and there exists a $\gamma(G)$-set D such that every vertex in D is adjacent to a vertex of degree one, then for any graph H with $\gamma(H) \geq 4$,*

$$\gamma_r(G \circ H) = 4\gamma(G).$$

Proof. Assume that $\gamma_t(G) = 2\gamma(G)$, $\gamma(H) \geq 4$ and let D be a $\gamma(G)$-set such that every vertex in D is adjacent to a vertex of degree one. We will show that $\gamma_r(G \circ H) \geq 4\gamma(G)$. Since $\gamma_t(G) = 2\gamma(G)$, the vertex set of G can be partitioned by the closed neighborhoods of vertices in D, *i.e.*, $V(G) = \cup_{x \in D} N[x]$ and $N[x] \cap N[y] = \emptyset$, for every $x, y \in D$, $x \neq y$. Now, let $f(W_0, W_1, W_2)$ be a $\gamma_r(G \circ H)$-function and let $x' \in N(x)$ be a vertex of degree one, for $x \in D$. Suppose that f assigns at most three legions to $N[x] \times V(H)$. We differentiate the following cases for the set $W = W_1 \cup W_2$.

1. *Case* $|W \cap (\{x\} \times V(H))| = 3$. Since $\gamma(H) \geq 4$, there exists at least one vertex in $\{x\} \times V(H)$ which is not dominated by the elements in W, which is a contradiction.

2. *Case* $|W_2 \cap (\{x\} \times V(H))| = 1$ *or* $|W_1 \cap (\{x\} \times V(H))| = 2$. In both cases there exists $y \in N(x)$ such that $|W_1 \cap (\{y\} \times V(H))| = 1$. Since $\gamma(H) \geq 4$, the movement of a legion from the vertex in $W_1 \cap (\{y\} \times V(H))$ to any vertex in $W_0 \cap (\{x\} \times V(H))$ produces unprotected vertices in $\{x\} \times V(H)$, which is a contradiction.

3. *Case* $|W_1 \cap (\{x\} \times V(H))| = 1$. Since $\gamma(H) \geq 4$, the movement of a legion from the vertex in $W_1 \cap (\{x\} \times V(H))$ to any vertex in $W_0 \cap (\{x'\} \times V(H))$ produces unprotected vertices in $\{x'\} \times V(H)$, which is a contradiction.

4. *Case* $|W \cap (\{x\} \times V(H)) + | = 0$. Since $\gamma(H) \geq 4$, there exists at least one vertex $(x', h) \in W_0$ which is not dominated by the elements in W, which is a contradiction.

According to the four cases above, for every $x \in D$ we have that f assigns at least four legions to $N[x] \times V(H)$, which implies that $\gamma_r(G \circ H) \geq 4\gamma(G)$.

Furthermore, by Corollary 18, $\gamma_r(G \circ H) \leq 4\gamma(G)$. Therefore, the result follows. □

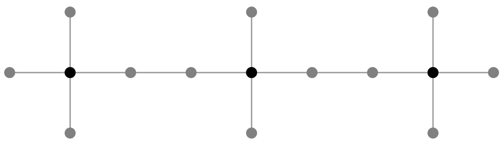

Figure 4.8: Example graph having $\gamma_t(G) = 2\gamma(G)$.

For the tree shown in Figure 4.8 we have $\gamma(T) = \rho(T) = 3$. Notice that the set of black-coloured vertices is the only dominating set of G which corresponds to the set of support vertices of T; is the only $\gamma(T)$-set and a $\rho(T)$-set. In this case $\gamma_t(G) = 2\gamma(G) = 6$. By Theorem 30, for any graph H with $\gamma(H) \geq 4$ we have $\gamma_r(G \circ H) = 12 = 4\gamma(G)$.

Corollary 31. *If the set of support vertices of a tree T is a $\rho(T)$-set, then for any graph H with $\gamma(H) \geq 4$,*

$$\gamma_r(T \circ H) = 4\gamma(T).$$

From Theorems 13 and 20 we have the following result.

Theorem 32. *If G is a graph such that $\gamma_{2,t}(G) = \max\{\gamma_r(G), 2\rho(G)\}$, then for any noncomplete graph H,*

$$\gamma_r(G \circ H) = \gamma_{2,t}(G).$$

According to Theorem 32, the problem of characterizing the graphs for which $\gamma_{2,t}(G) = \gamma_r(G)$ or $\gamma_{2,t}(G) = 2\rho(G)$ deserves being considered in future works.

Definition 16. *We will construct a family $\mathcal{H}_k$ of graphs such that $\gamma_r(G) = \gamma_{2,t}(G)$, for every $G \in \mathcal{H}_k$. A graph $G = (V, E)$ family $\mathcal{H}_k$ if and only if it is constructed from a cycle C_k and k empty graphs $N_{s_1}, \ldots, N_{s_k}$ of order $s_1, \ldots, s_k$, respectively, and joining by an edge each vertex from N_{s_i} with the vertices v_i and v_{i+1} of C_k. Here we are assuming that v_i is adjacent to v_{i+1} in C_k, where the subscripts are taken module k. Figure 4.9 shows a graph belonging to family $\mathcal{H}_k$, where $k = 4$, $s_1 = s_3 = 3$ and $s_2 = s_4 = 2$.*

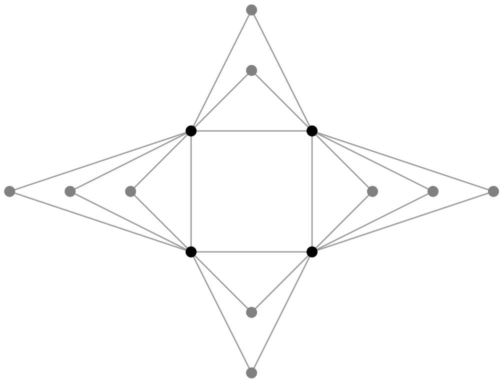

Figure 4.9: Example of family $\mathcal{H}_k$ defined

For any graph $G \in \mathcal{H}_k$ we have $\gamma_r(G) = \gamma_{2,t}(G) = k$. Therefore, by Theorem 32, for any $G \in \mathcal{H}_k$ and any graph H, $\gamma_r(G \circ H) = k$. We can see an example in Figure 4.9 where the set of balck-coloured vertices is a double dominating set:$\gamma_r(G) = \gamma_{2,t}(G) = 4$.

Definition 17. *From now on we say that a vertex $a \in V(H)$ satisfies Property $\mathcal{P}'$ if $\{a, b\}$ is a dominating set of H, for every $b \in V(H) \setminus N[a]$. In other words, $a \in V(H)$ satisfies Property $\mathcal{P}'$ if the subgraph induced by $V(H) \setminus N[a]$ is a clique.*

Proposition 33. *For any integer $n \geq 3$ and any noncomplete graph H,*

$$2 \leq \gamma_r(K_n \circ H) \leq 3.$$

Furthermore, $\gamma_r(K_n \circ H) = 2$ if and only if $\gamma_r(H) = 2$ or there exists a vertex of H which satisfies Property $\mathcal{P}'$.

Proof. By Remark 7 we have $\gamma_r(K_n \circ H) \geq 2$ and by Theorem 20 we have that $\gamma_r(K_n \circ H) \leq 3$.

To characterize the graphs with $\gamma_r(K_n \circ H) = 2$ we first assume that $\gamma_r(H) = 2$, and we will apply Remark 8 to the graph H. Let $u \in V(G)$ and let $\{a, b\} \subseteq V(H)$ which satisfy Property $\mathcal{P}$. We claim that the function $f(X_0, X_1, X_2)$ defined by $X_0 = V(K_n) \times V(H) \setminus \{(u, a), (u, b)\}$, $X_1 = \{(u, a), (u, b)\}$ and $X_2 = \emptyset$ is a $\gamma_r(K_n \circ H)$-function. To see this, we only need to observe that the movement of a legion from (u, a) (or from (u, b)) to a vertex in X_0 does not produce unprotected vertices. Now, if $\gamma(H) = 1$, then we define the $\gamma_r(K_n \circ H)$-function f by $X_0 = V(K_n) \times V(H) \setminus \{(u, z)\}$, $X_1 = \emptyset$ and $X_2 = \{(u, z)\}$, where $z \in V(H)$ is a vertex of maximum degree. On the other hand, if $a \in V(H)$ satisfies Property $\mathcal{P}'$, then we define the $\gamma_r(K_n \circ H)$-function f by $X_0 = (V(K_n) \times V(H)) \setminus \{(u_1, a), (u_2, a)\}$, $X_1 = \{(u_1, a), (u_2, a)\}$ and $X_2 = \emptyset$, where $u_1 \neq u_2$.

Conversely, assume that $\gamma_r(K_n \circ H) = 2$ and let $f(W_0, W_1, W_2)$ be a $\gamma_r(K_n \circ H)$-function. Notice that, $|W_1| + 2|W_2| = 2$. Now, if $W_2 = \{(u, a)\}$, then $\gamma(H) = 1$. From now on, assume that $W_1 = \{(u_1, a), (u_2, b)\}$ and $\gamma(H) \geq 2$. Assume that $u_1 = u_2$. In this case, $\{a, b\}$ is a dominating set of H and if there exists $x \in V(G) \setminus \{a, b\}$, then the movement of a legion from (u_1, a) to (u_1, x) does not produce unprotected vertices or the movement of a legion from (u_1, b) to (u_1, x) does not produce unprotected vertices, which implies that $\{a, b\}$ satisfy Property $\mathcal{P}$. Hence, by Remark 8, $\gamma_r(H) = 2$. Finally, if $u_1 \neq u_2$, then the movement of a legion from (u_2, a) to (u_1, c), where $c \in V(H) \setminus N[a]$, does not produce unprotected vertices, which implies that a satisfies Property $\mathcal{P}'$. □

Proposition 34. *Let H be a graph and let $n \geq 3$ be an integer. Then the following statements hold.*

(i) If $\gamma_r(H) \in \{2, 3\}$, then $\gamma_r(K_{1,n} \circ H) = \gamma_r(H)$.

(ii) If $\gamma_r(H) \geq 4$, then $3 \leq \gamma_r(K_{1,n} \circ H) \leq 4$.

(iii) If $\gamma(H) \geq 4$, then $\gamma_r(K_{1,n} \circ H) = 4$.

Proof. Let u_0 be a universal vertex of $K_{1,n}$. By Remark 7 we have that $\gamma_r(K_{1,n} \circ H) \geq 2$ and by Theorem 13, $\gamma_r(K_{1,n} \circ H) \leq 2\gamma_t(K_{1,n}) = 4$.

Let g be a $\gamma_r(H)$-function. Assume that $\gamma_r(H) \in \{2, 3\}$. The function $f : V(K_{1,n} \circ H) \longrightarrow \{0, 1, 2\}$ defined by $f(u_0, v) = g(v)$, for every $v \in V(H)$, and $f(u, v) = 0$, for every $u \in V(K_{1,n}) \setminus \{u_0\}$ and $v \in V(H)$, is a WRDF of $K_{1,n} \circ H$, which implies that $\gamma_r(K_{1,n} \circ H) \leq \gamma_r(H)$. Hence, if $\gamma_r(H) = 2$, then we are done. Since $n \geq 3$, for any $\gamma_r(K_{1,n} \circ H)$-function we have $f(H_{u_0}) \geq 2$ and, if $\gamma_r(H) \geq 3$, then $w(f) \geq 3$. Thus, (i) and (ii) follow.

Finally, if $\gamma(H) \geq 4$, then Theorem 30 leads to $\gamma_r(K_{1,n} \circ H) = 4$. □

We will now show that the bound given in Corollary 15 is tight. To this end, we need to introduce some additional notation.

Definition 18. *Given a graph G, let $\mathcal{P}_3(G)$ be the family of ordered sets $S = \{x_1, x_2, x_3\} \subset V(G)$ such that $\langle S \rangle \cong P_3$, $\delta(x_1) \geq 2$, $\delta(x_2) = 2$ and $\delta(x_3) = 1$.*

Lemma 35. *Let G and H be two graphs, and $\{x_1, x_2, x_3\} \in \mathcal{P}_3(G)$. If $\gamma(H) \geq 4$, then for any $\gamma_r(G \circ H)$-function f,*

$$\sum_{i=1}^{3} f(H_i) = 4.$$

Furthermore, there exists a $\gamma_r(G \circ H)$-function f, such that $f(H_2) = 2$ and $f(H_3) = 0$.

Proof. Suppose that there exists a $\gamma_r(G \circ H)$-function f with

$$\sum_{i=1}^{3} f(H_i) \leq 3.$$

We differentiate the following cases according to the value of $f(H_1)$.

1. $f(H_1) = 0$. If $f(H_2) = 0$ (resp. $f(H_3) = 0$), then there is an unprotected vertex in H_3 (resp. H_2). If $f(H_2) = 1$ (resp. $f(H_3) = 1$), then the movement of the legion from H_2 to H_3 (resp. from H_3 to H_2) produces an unprotected vertex in H_3 (resp. from H_2).

2. $f(H_1) = 1$. If $f(H_2) = 0$, then there is an unprotected vertex in H_3. If $f(H_2) = 1$, then the movement of the legion from H_2 to H_3 produces an unprotected vertex in H_3. Finally, If $f(H_2) = 2$, then the movement of the legion from H_1 to H_2 produces an unprotected vertex in H_2.

3. $f(H_1) = 2$. If $f(H_2) = 0$, then there is an unprotected vertex in H_3. If $f(H_2) = 1$, then the movement of the legion from H_2 to H_3 produces an unprotected vertex in H_3.

4. $f(H_1) = 3$. In this case the vertices in H_3 are unprotected.

In all cases above we obtain a contradiction, which implies that $f(H_1)+f(H_2)+f(H_3) \geq 4$. To conclude the proof we only need to observe that we can construct a $\gamma_r(G \circ H)$-function f with $f(H_1) + f(H_2) + f(H_3) = 4$, as we can take $f(H_1) = f(H_2) = 2$ and $f(H_3) = 0$. $\square$

We will now prove that there exists a family of trees T_n, which we will call combs, such that for any graph H with $\gamma(H) \geq 4$, $\gamma_r(T_n \circ H) = 2\left\lfloor \frac{2n}{3} \right\rfloor$. With this end we will now describe this family.

Definition 19. *We define comb family, T_n, as a family of trees such that taking a path P_k of length $k = \lceil \frac{n}{3} \rceil$, with vertices $v_1, \ldots, v_k$, and attach a path P_3 to each vertex $v_1, \ldots, v_{k-1}$, by identifying each v_i with a leaf of its corresponding copy of P_3. Finally, we attach a path of length $r = n - 3\lceil \frac{n}{3} \rceil + 2$ to v_k. Notice that*

$$n - 3\left\lceil \frac{n}{3} \right\rceil + 2 = \begin{cases} 0 \ \textit{if } n \equiv 1 \,(\textit{mod } 3); \\ 1 \ \textit{if } n \equiv 2 \,(\textit{mod } 3); \\ 2 \ \textit{if } n \equiv 0 \,(\textit{mod } 3). \end{cases}$$

Figure 4.10 shows the construction of T_n for different values of n. Notice that the comb of order six is simply $T_6 \cong P_6$.

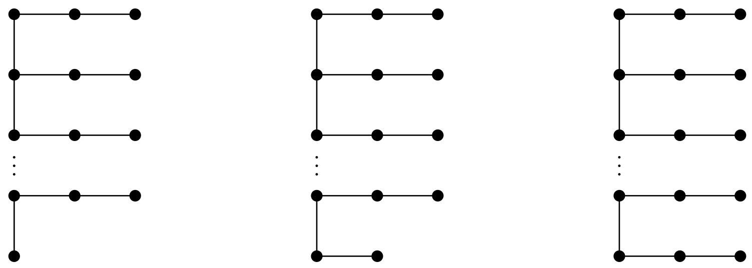

Figure 4.10: Example of construction of family of combs T_n for $r = 1, 2, 0$.

Proposition 36. *For any $n \geq 4$ and any graph H with $\gamma(H) \geq 4$,*

$$\gamma_r(T_n \circ H) = 2\left\lfloor \frac{2n}{3} \right\rfloor.$$

Proof. By Corollary 15 we have $\gamma_r(T_n \circ H) \leq 2\left\lfloor \frac{2n}{3} \right\rfloor$. In order to show that $\gamma_r(T_n \circ H) \geq 2\left\lfloor \frac{2n}{3} \right\rfloor$ we differentiate three cases.

If $n = 3k$, then Lemma 35 leads to $\gamma_r(T_n \circ H) = 4k = 2\left\lfloor \frac{2n}{3} \right\rfloor$. Now, if $n = 3(k-1)+1$, then Lemma 35 leads to $\gamma_r(T_n \circ H) \geq 4(k-1) = 2\left\lfloor \frac{2n}{3} \right\rfloor$. Finally, if $n = 3(k-1)+2$, then Lemma 35 leads to $\gamma_r(T_n \circ H) \geq 4(k-1)+2 = 2\left\lfloor \frac{2n}{3} \right\rfloor$. □

Definition 20. *Given a graph G, let family $\mathcal{P}_4(G)$ be the family of ordered sets $S = \{x_1, x_2, x_3, x_4\} \subset V(G)$ such that $\langle S \rangle \cong P_4$, $\delta(x_1) \geq 2$, $\delta(x_2) = \delta(x_3) = 2$ and $\delta(x_4) \geq 2$.*

For any G such that family $\mathcal{P}_4(G) \neq \emptyset$ we define the family $\mathcal{O}_4(G)$ of graphs G^* constructed from G as follows. Let $S \in \mathcal{P}_4(G)$ such that $\langle S \rangle = P_4 = (x_1, x_2, x_3, x_4)$, $X = \{x_1x_2, x_2x_3, x_3x_4\}$, $X_1 = N(x_1) \setminus \{x_2\}$, $X_4 = N(x_4) \setminus \{x_3\}$ and $Y = \{ab :\ a \in X_1 \text{ and } b \in X_4\}$. The vertex set of G^* is $V(G^*) = V(G) \setminus S$ and the edge set is $E(G^*) = (E(G) \setminus X) \cup Y$.

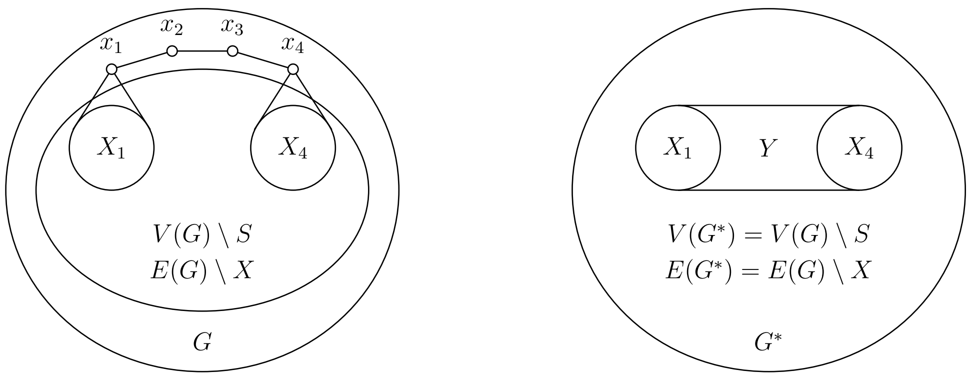

Figure 4.11: Schematic representation of the construction of graphs $G^* \in \mathcal{O}_4(G)$ from G.

Schematic representation of G and its corresponding G^* is shown in Figure 4.11. See remarked the neighborhoods of that $X_1 = N(x_1) \setminus \{x_2\}$ and $X_4 = N(x_4) \setminus \{x_3\}$. Notice that the resulting graph G^* is the result of contracting the edges X of G.

Theorem 37. *Let G be a graph such that family $\mathcal{P}_4(G) \neq \emptyset$ and let H be a graph. If $\gamma(H) \geq 4$, then for any $G^* \in \mathcal{O}_4(G)$,*

$$\gamma_r(G \circ H) = \gamma_r(G^* \circ H) + 4.$$

Proof. We provide the demonstration of this result in Annex 1. □

A simple case analysis shows that for $n \in \{3, 4, 5, 6\}$ and any graph H such that $\gamma(H) \geq 4$ we have $\gamma_r(C_n \circ H) = n$. Hence, Theorem 37 immediately leads to the following corollary.

Corollary 38. *Let $n \geq 3$ be an integer and let H be a graph. If $\gamma(H) \geq 4$, then*

$$\gamma_r(C_n \circ H) = n.$$

It is readily seen that if $\gamma(H) \geq 4$, then $\gamma_r(P_2 \circ H) = \gamma_r(P_3 \circ H) = \gamma_r(P_4 \circ H) = 4$ and $\gamma_r(P_5 \circ H) = 6$. Therefore, Theorem 37 leads to the following result.

Corollary 39. *Let $n \geq 2$ be an integer and let H be a graph. If $\gamma(H) \geq 4$, then*

$$\gamma_r(P_n \circ H) = \begin{cases} n, & n \equiv 0 \pmod 4; \\ n+2, & n \equiv 2 \pmod 4; \\ n+1, & \textit{otherwise}. \end{cases}$$

Chapter 5

On secure domination

This Section is devoted to obtain general bounds on $\gamma_r(G)$ and $\gamma_s(G)$ in terms of several invariants of G. As a consequence of the study we derive new inequalities of Nordhaus-Gaddum type involving secure domination and weak Roman domination. Later, in Subsection 5.1 the study is restricted to the particular case of Cartesian product graphs.

Theorem 40. [16] *Let $G \not\cong C_5$ be a connected graph. If $\delta(G) \geq 2$, then*

$$\gamma_s(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor.$$

An example of a graph with $\delta(G) = 3$ and $\gamma_s(G) = \gamma_2(G) = \left\lfloor \frac{n(G)}{2} \right\rfloor$ is the 3-cube graph. Notice that from the result above and the fact that $\gamma_r(G) \leq \gamma_s(G)$ we can conclude that if $G \not\cong C_5$ is connected and $\delta(G) \geq 2$, then $\gamma_r(G) \leq \left\lfloor \dfrac{n(G)}{2} \right\rfloor$.

With the aim of providing a general upper bound on the weak Roman domination number of any graph in terms of $n(G)$, we need to introduce some additional notation.

Definition 21. *For any support vertex v of a tree T, the set of leaves adjacent to v in T will be denoted by $L_T(v)$. Let $S(T)$ be the set of support vertices $v \in V(T)$ of degree $\delta(v) \leq |L_T(v)|+1$ and define*

$$X(T) = \bigcup_{v \in S(T)} (\{v\} \cup L_T(v)).$$

Let $T_0, T_1, \ldots, T_k$ be the sequence of all embedded subtrees of T, of order greater than or equal to three, defined as follows: $T_0 = T$ and T_i is the subtree of T_{i-1} induced by $V(T_{i-1}) \setminus X(T_{i-1})$, for every $i \in \{1, \ldots, k\}$. Notice that the smallest subtree T_k satisfies $|V(T_k) \setminus X(T_k)| \leq 2$.

With this notation in mind we proceed to prove the two following results.

Theorem 41. *For any connected nontrivial graph G,*

$$\gamma_r(G) \leq \left\lfloor \frac{2n(G)}{3} \right\rfloor .$$

Proof. Since the case $n(G) = 2$ is straightforward, we can assume that $n(G) \geq 3$. Let T be a spanning tree of G and $T_0, T_1, \ldots, T_k$ the sequence of all embedded subtrees of T of order greater than or equal to three defined previously. By Proposition 4, $\gamma_r(G) \leq \gamma_r(T)$. It remains to show that $\gamma_r(T) \leq \frac{2n(G)}{3}$. To this end, we proceed to construct a WRDF f such that $w(f) \leq \frac{2n(G)}{3}$.

For every $v \in X(T_i)$ and $i \in \{0, \ldots, k\}$ we set

$$f(v) = \begin{cases} 2 & \text{if } v \in S(T_i) \text{ and } |L_{T_i}(v)| \geq 2, \\ \\ 1 & \text{if } v \in S(T_i) \text{ and } |L_{T_i}(v)| = 1, \\ \\ 0 & \text{if } v \in X(T_i) \setminus S(T_i). \end{cases}$$

Notice that $V(G) = \bigcup_{i=0}^{k} X(T_i) \cup (V(T_k) \setminus X(T_k))$ and $X(T_i) \cap X(T_j) = \emptyset$ for every $i \neq j$. Hence, it remains to define $f(x)$ for every $x \in V(T_k) \setminus X(T_k)$, if any. Notice that for any $i \in \{0, \ldots, k\}$,

$$\sum_{v \in X(T_i)} f(v) = \sum_{v \in S(T_i)} f(v) \leq \frac{2}{3}|X(T_i)| \tag{5.1}$$

and, if there is a support vertex v of T_i with $|L_{T_i}(v)| = 1$, then

$$\sum_{v \in X(T_i)} f(v) = \sum_{v \in S(T_i)} f(v) < \frac{2}{3}|X(T_i)|. \tag{5.2}$$

Hence, if $V(T_k) = X_k$ then $\sum_{i=0}^{k} |X(T_i)| = n(G)$, which implies that

$$w(f) = \sum_{i=0}^{k} \left(\sum_{v \in X(T_i)} f(v) \right) \leq \frac{2}{3} \sum_{i=0}^{k} |X(T_i)| \leq \frac{2n(G)}{3}.$$

Suppose that $V(T_k) \setminus X_k = \{x\}$. In this case, we set $f(x) = 0$ whenever $f(v) = 2$ for some neighbour v of x, otherwise we set $f(x) = 1$. Obviously, if $f(x) = 0$, then

$$w(f) = \sum_{i=0}^{k} \left(\sum_{v \in X(T_i)} f(v) \right) + f(x) \leq \frac{2}{3} \sum_{i=0}^{k} |X(T_i)| \leq \frac{2(n(G)-1)}{3} < \frac{2n(G)}{3}.$$

Now, if $f(x)=1$, then (5.2) leads to $\sum_{v\in X_k} f(v) \leq \frac{2}{3}|X_k|-1$, which implies that

$$\begin{aligned}
w(f) =& \sum_{i=0}^{k-1}\left(\sum_{v\in X(T_i)} f(v)\right) + \sum_{v\in X_k} f(v) + f(x)\\
\leq& \frac{2}{3}\sum_{i=0}^{k-1}|X(T_i)| + \left(\frac{2}{3}|X(T_k)|-1\right)+1\\
=& \frac{2(n(G)-1)}{3} < \frac{2n(G)}{3}.
\end{aligned}$$

Finally, if $V(T_k)\setminus X_k=\{a,b\}$, then we set $f(a)=0$ and $f(b)=1$. Thus,

$$\begin{aligned}
w(f) &= \sum_{i=0}^{k}\left(\sum_{v\in X(T_i)} f(v)\right) + f(a)+f(b)\\
&\leq \frac{2}{3}\sum_{i=0}^{k}|X(T_i)|+1\\
&= \frac{2(n(G)-2)}{3}+1 < \frac{2n(G)}{3}.
\end{aligned}$$

In summary, we can conclude that $w(f)\leq \frac{2n(G)}{3}$, and it is readily seen that f is a WRDF. Therefore, the result follows. □

To see that the bound above is tight we can take any graph G_1 and construct the corona product of graphs $G\cong G_1\odot N_2$ by considering one copy of G_1 and $n(G_1)$ copies of N_2 and joining, by an edge, each vertex of G_1 with the vertices in the corresponding copy of N_2. In this case we have $\gamma_r(G)=2n(G_1)$ and $n(G)=3n(G_1)$.

Theorem 42. *Let T be a spanning tree of a connected graph G such that $n(G)\geq 3$. If $T_0,T_1,\ldots,T_k$ is the sequence of all embedded subtrees of T of order greater than or equal to three defined above, then*

$$\gamma_s(G)\leq \sum_{i=0}^{k}\sum_{v\in S(T_i)}|L_{T_i}(v)|+\varrho(T),$$

where $\varrho(T)=0$ if $V(T_k)=X(T_k)$ and $\varrho(T)=1$ otherwise.

Proof. Notice that Proposition 4 leads to $\gamma_s(G)\leq\gamma_s(T)$. Let

$$W=\bigcup_{i=0}^{k}\left(\bigcup_{v\in S(T_i)} L_{T_i}(v)\right)\cup W_k,$$

where W_k is defined as follows. If $V(T_k) = X(T_k)$, then we set $W_k = \emptyset$, otherwise we fix $x_k \in V(T_k) \setminus X(T_k)$ and we set $W_k = \{x_k\}$. To conclude that W is a secure dominating set for T we only need to observe that W is a dominating set and the movement of a guard from $L_{T_i}(v)$ to v does not produce undefended vertices, as well as, the movement of a guard from x_k to a vertex in $V(T_k) \setminus X(T_k)$ (if any) does not produce undefended vertices. Therefore, the result follows. □

The bound above is achieved, for instance, by the family of corona product graphs $G \cong G_1 \odot N_t$. Obviously, for any spanning tree T of G we have $\varrho(T) = 0$ and $tn(G_1) \leq \gamma_r(G) = \sum_{i=0}^{k} \sum_{v \in S(T_i)} |L_{T_i}(v)| + \varrho(T) = tn(G_1)$. Notice that the lower bound $\gamma_r(G) \geq tn(G_1)$ is deduced from the fact that every secure dominating set contains at least one guard per each vertex of degree one in G. In general, we can state the following tight bound in terms of the number of vertices of degree one, denoted by $\ell(G)$.

Remark 43. *For any graph G,*

$$\gamma_s(G) \geq \ell(G).$$

In particular, for any graph G',

$$\gamma_s(G' \odot N_t) = \ell(G' \odot N_t) = n(G')t.$$

Two edges in a graph G are independent if they are not adjacent in G. The matching number $\alpha'(G)$ of graph G, sometimes known as the edge independence number, is the cardinality of a maximum independent edge set.

Theorem 44. [18] *If a graph G does not have isolated vertices, then*

$$\gamma_s(G) \leq n(G) - \alpha'(G).$$

It is known that for every graph G with no isolated vertex $\alpha'(G) \geq \gamma(G)$ [27]. Hence, Theorem 44 leads to the following corollary.

Corollary 45. *If a graph G does not have isolated vertices, then*

$$\gamma_s(G) \leq n(G) - \gamma(G).$$

Recall that a graph without isolated vertices satisfies $\gamma(G) = n(G)/2$ if and only if its components are isomorphic to C_4 or to corona product graphs of the form $H \odot K_1$. If $\gamma(G) = n(G)/2$, then Corollary 45 leads to $\frac{n(G)}{2} = \gamma(G) \leq \gamma_r(G) \leq \gamma_s(G) \leq \frac{n(G)}{2}$. Thus, we deduce the following result.

Remark 46. *If $\gamma(G) = \frac{n(G)}{2}$, then $\gamma_r(G) = \gamma_s(G) = \frac{n(G)}{2}$.*

As we will show in Theorem 48, in some cases the bound provided by Theorem 44 can be improved. To this end, we need to introduce some additional notation.

Definition 22. *Let $\mathcal{D}(G)$ be the set of all $\gamma(G)$-sets. For every $S \in \mathcal{D}(G)$ we define*

$$T(S) = \{v \in V(G) \setminus S : \ N[v] = N[s] \text{ for some } s \in S\}.$$

Finally, we define the maximum twin set as

$$\tau(G) = \max\{|T(S)| : \ S \in \mathcal{D}(G)\}.$$

Recall that two vertices u, v are called true twins if $N[u] = N[v]$.

Lemma 47. *Let G be a graph such that no component of G is a complete graph. If S is a $\gamma(G)$-set, then $V(G) \setminus (S \cup T(S))$ is a dominating set.*

Proof. Since every vertex in $T(S)$ has a true twin in S, we only need to show that every vertex in S has a neighbour in $S' = V(G)\setminus(S\cup T(S))$. Notice that, since G has no isolated vertices and S is a $\gamma(G)$-set, every vertex in S has at least one neighbour outside of S. Suppose that there exists $s \in S$ such that $N(s)\cap S' = \emptyset$. In such a case, $N(s)\cap T(S) \neq \emptyset$ and, if $N(s)\cap S = \emptyset$, then the subgraph induced by $N[s]$ is a component of G, which is a contradiction. Thus, $N(s)\cap S \neq \emptyset$. Now, let $x \in N(s) \cap T(S)$. If s and x are true twins, then every neighbour of s belonging to S is a neighbour of x, while if s and x are not true twins, then there exists $s'' \in S \setminus \{s\}$ which is *twin* with x. Therefore, $S \setminus \{s\}$ is a dominating set, which is a contradiction. $\square$

Theorem 48. *If no component of G is a complete graph, then*

$$\gamma_s(G) \leq n(G) - \gamma(G) - \tau(G).$$

Proof. Let S be a $\gamma(G)$-set such that $|T(S)| = \tau(G)$. We will show that $S' = V(G)\setminus(S\cup T(S))$ is a secure dominating set. We already know from Lemma 47 that S' is a dominating set. It remains to show that for every $v \in S \cup T(S)$ there exists $u \in S' \cap N(v)$ such that $S'_{uv} = (S' \setminus \{u\}) \cup \{v\}$ is a dominating set. To this end, for every $u \in S'$ we define $P(u)$ as follows:

$$P(u) = \{v \in S : \ N(v) \cap S' = \{u\}\}.$$

If there exists $u \in S'$ such that $|P(u)| \geq 2$, then $S_1 = (S \setminus P(u)) \cup \{u\}$ is a dominating set and $|S_1| < |S| = \gamma(G)$, which is a contradiction. Hence, $|P(u)| \leq 1$ for every $u \in S'$. With this fact in mind, we differentiate two cases for $v \in V(G) \setminus S'$.

(i) $v \in S$. Suppose that $P(u) = \{v\}$ for some $u \in S'$. In this case, for every $w \in N(u) \cap (S \setminus \{v\})$ we have $|N(w) \cap S'| \geq 2$. So that, if there exists $y \in (N(u) \cap T(S)) \setminus N(v)$, then $|N(y) \cap S'| \geq 2$, as y has a *twin* in $S \setminus \{v\}$. Hence, S'_{uv} is a dominating set. From now on we assume that $|N(v) \cap S'| \geq 2$. Now, if there exists $u' \in N(v) \cap S'$ such that $P(u') = \emptyset$, then $|N(w) \cap S'| \geq 2$ for every $w \in N(u') \cap (S \setminus \{v\})$, and also for every $w \in (N(u') \cap T(S)) \setminus N(v)$, which implies $S'_{u'v}$ is a dominating set. Finally, suppose that $P(u) \neq \emptyset$ for every $u \in N(v) \cap S'$. Let

$$X = \{v\} \cup \left(\bigcup_{u \in N(v) \cap S'} P(u) \right).$$

Notice that $|X| = 1 + |N(v) \cap S'|$. Hence, $S_2 = (S \setminus X) \cup (N(v) \cap S')$ is a dominating set of G and $|S_2| < |S|$, which is a contradiction.

(ii) $v \in T(S)$. Let $v' \in S$ such that $N[v] = N[v']$. As discussed in *(i)*, there exists $u \in S'$ such that $S'_{uv'}$ is a dominating set. Since v and v' are true twins, we can conclude that S'_{uv} is also a dominating set.

According to the two cases above, S' is a secure dominating set. Therefore, $\gamma_s(G) \leq |S'| = n(G) - \gamma(G) - \tau(G)$. □

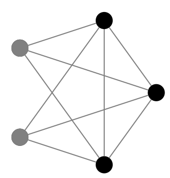

Figure 5.1: Example of Theorem 44 where true twins vertices are remarked.

Graph shown in Figure 5.1 shows an example where Theorem 48 improves the bound given by Theorem 44, we take the graph $G \cong K_3 + N_2 \cong K_5 - e$. In this case $\gamma(G) = 1$, $\tau(G) = 2$ and $\alpha'(G) = 2$, which implies that $\gamma_s(G) \leq n(G) - \gamma(G) - \tau(G) = 2 < 3 = n(G) - \alpha'(G)$.

It is well known that for any graph G, $\gamma(G) \geq \rho(G)$, [27]. Meir and Moon [34] showed in 1975 that $\gamma(T) = \rho(T)$ for any tree T. We remark that in general, these $\gamma(T)$-sets and $\rho(T)$-sets are not identical. The following result is a direct consequence of Theorem 48.

Corollary 49. *If no component of G is a complete graph, then*

$$\gamma_s(G) \leq n(G) - \rho(G) - \tau(G).$$

To see the sharpness of the bound above, consider the corona product graph $G_1 \odot N_p$, where G_1 is an arbitrary graph. In this case, $n(G_1 \odot N_p) = n(G_1)(p+1)$, $\rho(G_1 \odot N_p) = n(G_1)$ and $\gamma_s(G_1 \odot N_p) = n(G_1)p = n(G_1 \odot N_p) - \rho(G_1 \odot N_p) = n(G_1 \odot N_p) - \gamma(G_1 \odot N_p)$. From $G' \cong G_1 \odot N_2$ we can construct a family of graphs G of order $n(G) = 3n(G_1) + l_1 + \cdots + l_{n(G_1)}$ with $\gamma_s(G) = n(G) - \gamma(G) - \tau(G)$. We construct G from G' and a $\gamma(G')$-set $S = \{v_1, \ldots, v_{n(G_1)}\}$ by replacing every $v_j \in S$ with a copy of K_{l_j} and joining by an edge each vertex of K_{l_j} with each neighbour of v_j in G'.

As shown in [40], the domination number of any graph G is bounded below by $\frac{n(G)}{\Delta(G)+1}$. Therefore, the following result is deduced from Theorem 48.

Corollary 50. *If no component of G is a complete graph, then*

$$\gamma_s(G) \leq \left\lfloor \frac{n(G)\Delta(G)}{\Delta(G)+1} \right\rfloor - \tau(G).$$

The bound above is tight. For instance, it is achieved for any graph isomorphic to $K_n - e$. In this case $\tau(G) = n(G) - 2$ and $\Delta(G) = n(G) - 1$ so $\gamma_s(G) = 2$.

Since $\gamma_r(G) \leq 2\gamma(G)$ and $\gamma_r(G) \leq \gamma_s(G)$, Theorem 48 leads to the following upper bounds on the weak Roman domination number.

Corollary 51. *If no component of G is a complete graph, then the following assertions hold.*

(i) $\gamma_r(G) \leq \left\lfloor \dfrac{n(G) + \gamma(G) - \tau(G)}{2} \right\rfloor$.

(ii) If $\gamma(G) \geq \frac{n(G)}{3}$, *then* $\gamma_r(G) \leq 2\gamma(G) - \tau(G)$.

To see the sharpness of the bounds above, consider the corona graph $G \cong G_1 \odot N_2$, where G_1 is an arbitrary graph. In this case, $n(G) = 3n(G_1)$, $\gamma(G) = n(G_1)$, $\tau(G) = 0$ and $\gamma_r(G) = 2n(G_1)$. Another example of equality for bound (i) is $G \cong K_n - e$, where $\gamma_r(G) = 2$, $\tau(G) = n(G) - 3$ and $\gamma(G) = 1$.

The minimum number of cliques of a given graph G needed to cover the vertex set $V(G)$ is called the clique covering number of G and denoted by $\theta(G)$. Before stating our next result we need to recall the following theorem, which states a Nordhaus-Gaddum inequality for the chromatic number of a graph.

Theorem 52. [13] *For any graph G,*

$$\chi(G) + \chi(\overline{G}) \leq n(G) + 1 \text{ and } \chi(G)\chi(\overline{G}) \leq \frac{(n(G)+1)^2}{4}.$$

Theorem 53. *The following statements hold for any graph G.*

(i) $\gamma_s(G) \leq \theta(G)$.

(ii) $\gamma_r(G) + \gamma_r(\overline{G}) \leq \gamma_s(G) + \gamma_s(\overline{G}) \leq n(G) + 1$.

(iii) $\gamma_r(G)\gamma_r(\overline{G}) \leq \gamma_s(G)\gamma_s(\overline{G}) \leq \dfrac{(n(G)+1)^2}{4}$.

Furthermore, if $G \not\cong C_5$ is a connected graph with $\delta(G) \geq 2$ and $\Delta(G) \leq n(G) - 3$, then the following statement hold.

(iv) $\gamma_r(G) + \gamma_r(\overline{G}) \leq \gamma_s(G) + \gamma_s(\overline{G}) \leq n(G) - 1$ *for $n(G)$ odd and*
$\gamma_r(G) + \gamma_r(\overline{G}) \leq \gamma_s(G) + \gamma_s(\overline{G}) \leq n(G)$ *for $n(G)$ even.*

(v) $\gamma_r(G)\gamma_r(\overline{G}) \leq \gamma_s(G)\gamma_s(\overline{G}) \leq \frac{(n(G)-1)^2}{4}$ *for $n(G)$ odd and*
$\gamma_r(G)\gamma_r(\overline{G}) \leq \gamma_s(G)\gamma_s(\overline{G}) \leq \frac{(n(G))^2}{4}$ *for $n(G)$ even.*

Proof. Let Π be a partition of $V(G)$ into cliques such that $|\Pi| = \theta(G)$. The proof of *(i)* directly follows from the fact that any set formed by one representative of each clique in Π is a secure dominating set.

Since $\chi(G) = \theta(\overline{G})$, (i) and Theorem 52 lead to

$$\gamma_s(G) + \gamma_s(\overline{G}) \leq \theta(G) + \theta(\overline{G}) = \chi(\overline{G}) + \chi(G) \leq n(G) + 1$$

and

$$\gamma_s(G)\gamma_s(\overline{G}) \leq \theta(G)\theta(\overline{G}) = \chi(\overline{G})\chi(G) \leq \frac{(n(G)+1)^2}{4},$$

as required. Finally, *(iv)* and *(v)* are a direct consequence of Theorem 40. □

The inequalities above are tight. For instance, *(i)* is achieved by the graphs shown in Figure 5.2, *(ii)* and *(iii)* are achieved by the self-complementary graph shown in Figure 5.2 (on the left) and also by C_5. In both cases we have $n(G) = 5$ and $\gamma_r(G) = \gamma_s(G) = 3$. Finally, *(iv)* and *(v)* are achieved by the self-complementary graph shown in Figure 5.2 (on the right), in this case we have $n(G) = 8$ and $\gamma_r(G) = \gamma_s(G) = 4$.

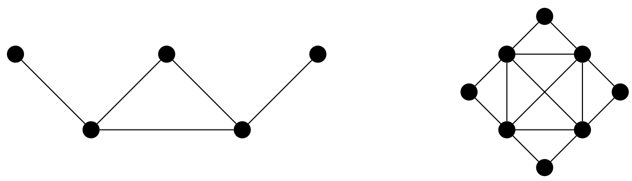

Figure 5.2: Two self-complementary graphs.

5.1 Cartesian product graphs

This Section focuses on the results obtained for secure domination on Cartesian product graphs. This product has been extensively investigated from various perspectives. For instance, the most popular open problem in the area of domination theory is known as Vizing's conjecture. Vizing [39] suggested that for any graphs G and H,

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H).$$

Several researchers have worked on it, for instance, some partial results appears in [15, 25]. The study of the secure domination number of Cartesian product graphs was initiated by Cockayne et al. in [19], where they obtained bounds on $\gamma_s(C_k\Box C_t)$ and $\gamma_s(P_k\Box P_t)$ in terms of k and t.

Before stating our first result we need to recall the following well known lower bound on the domination number of any Cartesian product graph.

Lemma 54. [24] *For any pair of graphs G and H,*

$$\gamma(G\Box H) \geq \min\{n(G), n(H)\}.$$

Theorem 55. *For any graphs G and H, the following statements hold.*

(i) $\min\{n(G), n(H)\} \leq \gamma_r(G\Box H) \leq \min\{n(G)\gamma_r(H), n(H)\gamma_r(G)\}$.

(ii) $\min\{n(G), n(H)\} \leq \gamma_s(G\Box H) \leq \min\{n(G)\gamma_s(H), n(H)\gamma_s(G)\}$.

Proof. Let $f(U_0, U_1, U_2)$ be a $\gamma_r(G)$-function. In order to prove the upper bound, we claim that the function $g : V(G\Box H) \longrightarrow \{0, 1, 2\}$ defined by $g(x, y) = f(x)$ is a WRDF on $G\Box H$, where

$$\{W_0 = U_0 \times V(H), W_1 = U_1 \times V(H), W_2 = U_2 \times V(H)\}$$

is the partition of $V(G\Box H)$ associated to g. To see this we only need to observe the following two facts.

(a) Since every $x \in U_0$ is dominated by some $x' \in U_1 \cup U_2$, every $(x, y) \in W_0$ is dominated by $(x', y) \in W_1 \cup W_2$.

(b) Since for every $x \in U_0$ there exists $x' \in N(x) \cap (U_1 \cup U_2)$ such that the movement of a guard from x' to x does not produce undefended vertices in G, the movement of a guard from $(x', y) \in W_1 \cup W_2$ to $(x, y) \in W_0$ does not produce undefended vertices in the subgraph of $G\Box H$ induced by $V(G) \times \{y\}$, which is isomorphic to G.

According to Facts *(a)* and *(b)* we can conclude that g is a WRDF on $G\Box H$, which implies that $\gamma_r(G\Box H) \leq w(g) = n(H)w(f) = n(H)\gamma_r(G)$, as required. By analogy we deduce that $\gamma_r(G\Box H) \leq n(G)\gamma_r(H)$. Therefore, the upper bound of *(i)* follows. The proof of the upper bound of *(ii)* is deduced by analogy to the previous one by taking a WRDF $f(U_0, U_1, U_2)$ such that $U_2 = \emptyset$ and $|U_1| = \gamma_s(G)$. Finally, the lower bounds are deduced from Lemma 54, as $\gamma_s(G\Box H) \geq \gamma_r(G\Box H) \geq \gamma(G\Box H) \geq \min\{n(G), n(H)\}$. □

As we show in the following results, the bounds above are tight.

Corollary 56. *Let t be an integer. If $2 \leq n(H) \leq t$, then $\gamma_r(K_t\Box H) = \gamma_s(K_t\Box H) = n(H)$.*

According to this result, it remains to study the weak Roman domination number and the secure domination number of $K_t\Box H$ for $n(H) > t$. Our next result covers two particular cases.

Proposition 57. *For any integers $t \geq 3$ and $t' \geq 3$,*

$$\gamma_r(K_t\Box C_{t'}) = \gamma_r(K_t\Box P_{t'}) = \gamma_s(K_t\Box P_{t'}) = \gamma_s(K_t\Box C_{t'}) = t'.$$

Proof. By Theorem 55 and Propositions 2 and 4 we have that

$$\min\{t, t'\} \leq \gamma_r(K_t\Box C_{t'}) \leq \gamma_r(K_t\Box P_{t'}) \leq t'$$

and

$$\min\{t, t'\} \leq \gamma_r(K_t\Box C_{t'}) \leq \gamma_s(K_t\Box C_{t'}) \leq \gamma_s(K_t\Box P_{t'}) \leq t'.$$

It remains to show that $\gamma_r(K_t\Box C_{t'}) \geq t'$ for $t' > t \geq 3$. Let $f(W_0, W_1, W_2)$ be a $\gamma_r(K_t\Box C_{t'})$-function and $V(C_{t'}) = \{v_1, \ldots, v_{t'}\}$, where the subscripts are taken modulo t' and $v_iv_{i+1} \in E(C_{t'})$ for any $i \leq t'$. Let $A_i = (V(K_t) \times \{v_i\})$ and $\alpha_i = f(A_i)$ for every $i \in \{1, \ldots, t'\}$. We differentiate the following cases in which $\alpha_i = 0$ for some i. Symmetric cases are omitted.

(i) $\alpha_i = 0$. Since $W_1 \cup W_2$ is a dominating set, we can conclude that

$$\alpha_{i-1} + \alpha_i + \alpha_{i+1} \geq t \geq 3.$$

(ii) $\alpha_{i-1} = \alpha_{i+1} = 0$ and $\alpha_i = 1$. In this case, no guard can move from A_i to A_{i+1} (or to A_{i-1}), which implies that $\alpha_{i-2} \geq t$ and $\alpha_{i+2} \geq t$. Hence, we can conclude that

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} \geq t + 1 \geq 4 \text{ and } \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \alpha_{i+2} \geq 1 + t \geq 4.$$

In this case, if $t' \geq 6$, then

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \alpha_{i+2} \geq 2t + 1 \geq 7.$$

(iii) $\alpha_i = 2$ and $\alpha_{i-1} = \alpha_{i+1} = 0$. From *(i)* we know that $\alpha_{i-2} \geq t-2$ and $\alpha_{i+2} \geq t-2$. Suppose that $\alpha_{i-2} = t-2$ and $\alpha_{i+2} < t$. Notice that $W_2 \cap (A_{i-2} \cup A_i) = \emptyset$, as every vertex in A_{i-1} has to be dominated by some vertex in $W_1 \cup W_2$. Hence, for $(u, v_i), (u', v_i) \in V_1$ we have that $(u, v_{i-2}), (u', v_{i-2}) \in V_0$ and $(u, v_{i+2}) \in V_0$ or $(u', v_{i+2}) \in V_0$, as $\alpha_{i-2} = t-2$ and $\alpha_{i+2} < t$. We can assume that $(u, v_{i+2}) \in V_0$. Thus, the movement of a guard form (u, v_i) to (u, v_{i-1}) produces undefended vertices in A_{i+1}, which is a contradiction. Hence, $\alpha_{i-2} + \alpha_{i+2} \geq 2(t-1)$ and so we can conclude that

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} + \alpha_{i+2} \geq 2t \geq 6.$$

According to the conclusions derived from the cases above we can deduce that,

$$\gamma_r(K_t \Box C_{t'}) = w(f) = \sum_{i=1}^{t'} \alpha_i \geq t'.$$

Therefore, the result follows. □

Notice that the result above does not include the case of complete graphs of order two. For this case we propose the following conjecture.

Conjecture 58. *For any integer $t \geq 2$*

$$\gamma_s(P_t \Box K_2) = \left\lceil \frac{3t+1}{4} \right\rceil.$$

Furthermore, for $t \geq 3$,

$$\gamma_s(C_t \Box K_2) = \begin{cases} \left\lceil \frac{3t}{4} \right\rceil + 1, & \text{if } t \equiv 4 \pmod 8 \\ \left\lceil \frac{3t}{4} \right\rceil, & \text{otherwise.} \end{cases}$$

Regarding the conjecture above, we would emphasize that it is known from [32] that $\gamma(P_t \Box K_2) = \left\lceil \frac{t+1}{2} \right\rceil$ and from [20] that $\gamma_R(P_t \Box K_2) = t+1$.

Proposition 59. *Let $t \geq 2$ and $t' \geq 2$ be two integers. The following statements hold.*

(i) $\gamma_r(K_t \Box K_{1,t'-1}) = \min\{2t, t'\}$.

(ii) $\gamma_s(K_t \Box K_{1,t'-1}) = t'$.

Proof. From Theorem 55 we have that $\gamma_r(K_t\Box K_{1,t'-1}) \leq \min\{2t,t'\}$. We proceed to show that $\gamma_r(K_t\Box K_{1,t'-1}) \geq \min\{2t,t'\}$. Let $f(W_0,W_1,W_2)$ be a $\gamma_r(K_t\Box K_{1,t'-1})$-function and let y_0 be the universal vertex of $K_{1,t'-1}$. Suppose that $\gamma_r(K_t\Box K_{1,t'-1}) < \min\{2t,t'\}$. Now, since $\gamma_r(K_t\Box K_{1,t'-1}) < 2t$, there exists $x \in V(K_t)$ such that $f(\{x\}\times V(K_{1,t'-1})) \leq 1$ and, since $\gamma_r(K_t\Box K_{1,t'-1}) < t'$, there exist $y \in V(K_{1,t'-1})$ such that $V(K_t)\times\{y\} \subseteq W_0$. If $y = y_0$, then there is exactly one guard for each copy of K_t different from the one associated to y_0 (as every vertex has to be defended), which implies that the movement of any guard to a vertex in $V(K_t)\times\{y_0\}$ produces undefended vertices, so that $y \neq y_0$. Notice that $f(V(K_t)\times\{y_0\}) \geq t$, otherwise there are undefended vertices in $V(K_t)\times\{y\}$. Now, suppose that $V(K_t)\times\{y'\} \subseteq W_0$, for some $y' \in V(K_{1,t'-1})\setminus\{y_0,y\}$. In such a case, (x,y') and (x,y) are only defended by a guard located at (x,y_0), but (x,y) will become undefended after the movement of that guard to (x,y'), which is a contradiction. Hence, $\sum_{v\neq y_0} f(V(K_t)\times\{v\}) \geq t'-2$, and so $w(f) \geq t+t'-2 \geq t'$, which is a contradiction again. Thus, $\gamma_r(K_t\Box K_{1,t'-1}) \geq \min\{2t,t'\}$, as required. Therefore, *(i)* follows.

We now proceed to prove *(ii)*. As above, let y_0 be the universal vertex of $K_{1,t'-1}$, W a $\gamma_s(K_t\Box K_{1,t'-1})$-set and $u \in V(K_t)$. Suppose that $|W| \leq t'-1$. In such a case, there exists $v \in V(K_{1,t'-1})$ such that $W\cap(V(K_t)\times\{v\}) = \emptyset$. Notice that $N(u,v)\cap W \neq \emptyset$. We differentiate two cases.

(i') $v \neq y_0$. Since W is a dominating set, $V(K_t)\times\{y_0\} \subseteq W$. Thus, there exists $v_1 \in V(K_{1,t'-1})\setminus\{v,y_0\}$ such that $V(K_t)\times\{v_1\} \subseteq \overline{W}$. Hence, $N[(u,v)]\cap W = \{(u,y_0)\} = N[(u,v_1)]\cap W$, and so $(W\setminus\{(u,y_0)\})\cup\{(u,v_1)\}$ is not a dominating set, which is a contradiction.

(ii') $v = y_0$. Since W is a dominating set and $|W| < t'$, for every $v' \in V(K_{1,t'-1})\setminus\{y_0\}$ we have that $|(V(K_t)\times\{v'\})\cap W| = 1$. Hence, for $u \in V(K_t)$ such that $(u,v') \in W$ and $u' \in V(K_t)\setminus\{u\}$ we have that $N[(u',v')]\cap W = \{(u,v')\}$. Thus, for every $v' \in V(K_{1,t'-1})\setminus\{y_0\}$ and $u \in V(K_t)$ such that $(u,v') \in W$, we have that $(W\setminus\{(u,v')\})\cup\{(u,y_0)\}$ is not a dominating set, which is a contradiction.

According to the two cases above we can conclude that $\gamma_s(K_t\Box K_{1,t'-1}) = |W| \geq t'$. Finally, Theorem 55 leads to $\gamma_s(K_t\Box K_{1,t'-1}) = t'$. □

Proposition 60. *For any graph G and any integer $t > 2n(G) \geq 4$,*

$$\gamma_r(G\Box K_{1,t-1}) = 2n(G).$$

Proof. By Theorem 55 we have $\gamma_r(G\Box K_{1,t-1}) \leq 2n(G)$. To conclude the proof we only need to observe that Propositions 4 and 59 lead to $\gamma_r(G\Box K_{1,t-1}) \geq \gamma_r(K_{n(G)}\Box K_{1,t-1}) = 2n(G)$. □

Theorem 61. *If no component of a graph H is a complete graph, then for any nontrivial graph G,*

$$\gamma_s(G\Box H) \leq n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H) - \gamma(G)\tau(H).$$

Proof. In this proof we use the set $T(S)$ as defined in Definition 22. Let S_1 be a $\gamma(G)$-set and S_2 a $\gamma(H)$-set such that $|T(S_2)| = \tau(H)$. We will show that $W = (S_1 \times S'_2) \cup (\overline{S_1} \times S_2)$ is a secure dominating set of $G\Box H$, where $S'_2 = V(H) \setminus (S_2 \cup T(S_2))$. First of all, notice that W is a dominating set of $G\Box H$ as S_2 and S'_2 are dominating sets in H (by Lemma 47). We differentiate the following three cases for $(x, y) \in \overline{W}$.

(i) $(x, y) \in S_1 \times \overline{S'_2}$. In the proof of Theorem 48 we have shown that S'_2 is a secure dominating set. Hence, for each vertex $(x, y) \in S_1 \times \overline{S'_2}$ there exists $(x, y') \in S_1 \times S'_2$ such that the movement of a guard from (x, y') to (x, y) does not produce undefended vertices in $\{x\} \times \overline{S'_2}$. Such a movement of guards does not produce undefended vertices in $\overline{S_1} \times \{y'\}$, as these vertices are dominated by the ones in $\overline{S_1} \times S_2$.

(ii) the movement of a guard from (x, y') to (x, y) does not produce undefended vertices in $S_1 \times \{y'\}$, as these vertices are dominated by the ones in $S_1 \times \{y\}$. Such a movement of guards does not produce undefended vertices in $\{x\} \times S'_2$, as these vertices are dominated by the ones in $\{x'\} \times S'_2$, for every $x' \in S_1 \cap N(x)$. Now, suppose that $y'' \in N(y') \cap T(S_2)$. If $|N(y'') \cap S_2| \geq 2$, then (x, y'') remains defended after the above mentioned movement of guards. If $|N(y'') \cap S_2| = \{y'\}$, then y' and y'' are twins, which implies $(x, y'') \in N(x, y)$, so that (x, y'') remains defended after the movement of a guard form (x, y') to (x, y).

(iii) $(x, y) \in \overline{S_1} \times T(S_2)$. Let $y' \in S_2$ such that $N[y] = N[y']$. As in the previous case, the movement of a guard from (x, y') to (x, y) does not produce undefended vertices in $S_1 \times \{y'\}$. On the other hand, since y and y' are twins, the movement of a guard from (x, y') to (x, y) does not produce undefended vertices in $\{x\} \times \overline{S_2}$.

According to the three cases above, W is a secure dominating set of $G\Box H$. Therefore,

$$\gamma_s(G\Box H) \leq |W| = n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H) - \gamma(G)\tau(H)$$

as desired. □

According to the result above, for any noncomplete graph H,

$$\gamma_s(K_t\Box H) \leq (t-2)\gamma(H) + n(H) - \tau(H).$$

It is not difficult to check that the bound above is tight. For instance, it is achieved by $H \cong K_l + N_3$ for $l \geq 2$, as $\gamma_s(K_3\Box(K_l + N_3)) = 5$, $\gamma(H) = 1$ and $\tau(H) = l - 1$. Notice that, in this case, Theorem 61 gives a better result than Theorem 55.

We learned from Theorem 40 that $\gamma_s(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$ for every graph $G \not\cong C_5$ having minimum degree $\delta(G) \geq 2$. If G and H have no isolated vertices, then $\gamma(G) \in \{1, \ldots, \lfloor n(G)/2 \rfloor\}$ and $\gamma(H) \in \{1, \ldots, \lfloor n(H)/2 \rfloor\}$. Hence, we can state the following remark which shows that the bound provide by Theorem 61 is never worse that the bound $\gamma_s(G\Box H) \leq \left\lfloor \frac{n(G)n(H)}{2} \right\rfloor$ deduced from Theorem 40.

Remark 62. *If G and H have no isolated vertices, then*

$$n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H) \leq \left\lfloor \frac{n(G)n(H)}{2} \right\rfloor.$$

The inequality chain

$$\gamma_r(G\Box H) \leq \gamma_s(G\Box H) \leq n(G)\gamma(H) + n(H)\gamma(G) - 2\gamma(G)\gamma(H)$$

is tight. It is achieved for $P_3\Box P_3$ and $K_2\Box K_2 \cong C_4$, as $\gamma_r(P_3\Box P_3) = 4$ and $\gamma_r(C_4) = 2$. Proposition 63 provides another example of graphs for which this inequality chain is achieved.

Proposition 63. *For any integer $t \geq 3$,*

$$\gamma_r(K_{1,t-1}\Box K_{1,t-1}) = \gamma_s(K_{1,t-1}\Box K_{1,t-1}) = 2(t-1).$$

Proof. According to Theorem 61, we only need to prove the lower bound $\gamma_r(K_{1,t-1}\Box K_{1,t-1}) \geq 2(t-1)$. Let $f(W_0, W_1, W_2)$ be a $\gamma_r(K_{1,t-1}\Box K_{1,t-1})$-function and, for simplicity, set $V = V(K_{1,t-1})$. Let $x \in V$ be the vertex of degree $t-1$. From now on, we suppose that $w(f) \leq 2t-3$. We proceed to show the following claim.

Claim 1. $f(\{u\} \times V) \geq 1$, for every $u \in V \setminus \{x\}$.

In order to prove Claim 1, we suppose that there exists $u \in V\setminus\{x\}$ such that $f(\{u\}\times V) = 0$. In such a case, $f(x, y) \geq 1$, for every $y \in V$. Now, since $w(f) \leq 2t-3$, there exist $u' \in V\setminus\{x, u\}$ and $v \in V$ such that $f(\{u'\} \times V) = 0$ and $f(x, v) = 1$, which is a contradiction as (u', v) is undefended after the movement of the guard located in (x, v) to (u, v). Thus, Claim 1 follows.

Since $w(f) \leq 2t - 3$, Claim 1 leads to the following ones.

Claim 2. There exists $u^* \in V \setminus \{x\}$ such that $f(\{u^*\} \times V) = 1$.

Claim 3. There exists $v^* \in V \setminus \{x\}$ such that $f(x, v^*) = 0$.

We differentiate the following two cases for $f(u^*, x)$.

(i) $f(u^*, x) = 0$. By Claims 2 and 3 we can conclude that $f(u^*, v^*) = 1$, otherwise (u^*, v^*) is not dominated by the elements in $W_1 \cap W_2$. Since every vertex in $\{u^*\} \times V \setminus \{(u^*, x), (u^*, v^*)\}$ has to be dominated by some vertex in $W_1 \cup W_2$, from $w(f) \leq 2t - 3$ and Claim 1 we deduce that $f(x, v) = 1$ for every $v \in V \setminus \{x, v^*\}$, $f(\{u\} \times V) = 1$ for every $u \in V \setminus \{x, u^*\}$, and $f(x, x) = 0$. Hence, the movement of any guard from a vertex in $\{x\} \times V$ to (x, x) produces undefended vertices in $\{u^*\} \times V$, and the movement of a guard from a vertex of the form (a, x) to (x, x) leaves vertex (a, v^*) undefended. In both cases we have a contradiction.

(ii) $f(u^*, x) = 1$. In this case, (u^*, x) is the only vertex in $W_1 \cup W_2$ which is adjacent to (u^*, v^*). Hence, the movement of a guard from (u^*, x) to (u^*, v^*) does not produce undefended vertices, and so from $w(f) \leq 2t - 3$ and Claim 1 we deduce that $f(x, v) = 1$ for every $v \in V \setminus \{x, v^*\}$, $f(\{u\} \times V) = 1$ for every $u \in V \setminus \{x, u^*\}$, and $f(x, x) = 0$. Thus, the movement of a guard from a vertex of the form (a, x) to (x, x) leaves vertex (a, v^*) undefended, which is a contradiction.

According to the two cases above we can conclude that, $w(f) \geq 2(t - 1)$, as required. $\square$

As usual in domination theory, when studying a domination parameter, we can ask if a Vizing-like conjecture can be proved or formulated. By Proposition 63 we can claim that there are graphs with

$$\gamma_s(G \Box H) \not\geq \gamma_s(G)\gamma_s(H),$$

i.e., for any $p \geq 3$ we have $\gamma_s(K_{1,p} \Box K_{1,p}) = 2p < p^2 = \gamma_s(K_{1,p})\gamma_s(K_{1,p})$.

Theorem 64. *Let $f_H = (V_0, V_1, V_2)$ be a $\gamma_r(H)$-function of a graph H such that $V_2 \neq \emptyset$, and let $Y = V(H) \setminus N[V_2]$. For any graph G,*

$$\gamma_r(G \Box H) \leq 2n(G)|V_2| + |Y|\gamma_r(G).$$

Proof. Let $f_G = (U_0, U_1, U_2)$ be a $\gamma_r(G)$-function, $W_1 = U_1 \times Y$ and $W_2 = (V(G) \times V_2) \cup (U_2 \times Y)$. In order to show that $f = (W_0, W_1, W_2)$ is a WRDF of $G \Box H$, we differentiate the following two cases for $(x, y) \in W_0$.

(i) $(x, y) \in V(G) \times (N(V_2) \setminus V_2)$. Since there exists $y' \in V_2 \cap N(y)$, the movement of a guard from (x, y') to (x, y) does not produce undefended vertices.

(ii) $(x, y) \in U_0 \times Y$. Since f_G is a $\gamma_r(G)$-function, there exists $x' \in U_1 \cup U_2$ such that the movement of a guard from x' to x does not produce undefended vertices. Which implies that the movement of a guard from (x', y) to (x, y) does not produce undefended vertices in $V(G) \times Y$.

$\square$

Notice that for any graph with $\gamma_r(H) = 2\gamma(H)$, Theorems 55 and 64 lead to the same result $\gamma_r(G \Box H) \leq 2n(G)\gamma(H)$. In order to show an example where Theorem 64 gives a better result we take $G \cong K_3$ and the graph H shown in Figure 5.3. In this case, an optimum solution consists of two guards at each vertex of the copy of K_3 corresponding to the vertex $v \in V(H)$ of maximum degree and one guard at each copy of K_3 corresponding to the vertices of H nonajacent to v.

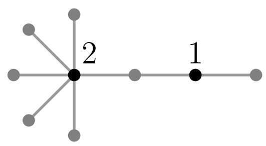

Figure 5.3: A graph with $\gamma_r(H) = 3$, $|Y| = 2$ and $\gamma_r(K_3 \Box H) = 2n(G)|V_2| + |Y|\gamma_r(G) = 8$.

Chapter 6

Conclusion

6.1 Lessons learned

After only a little research on the utilization of graphs, it was easy to realize how common they are and the amount of applications they have; a particular example is informatics where graphs are rapidly associated to a network structure. This amount and variety of possible applications accentuates the need of investing in research in order to take the maximum advantage of theory in real implementation.

Dominating sets and graph protection have a big impact on the way in which relations are treated among huge amounts of data. As lexicographic product is a product that increases the connection between nodes, the obtained graphs are very willing to be used in domination; which benefits the amount of relations in a graph. For the part of Cartesian product, this is the most natural and common product which contributes more on the usability of the results obtained. The difference between both specializations of the thesis is that lexicographic product has been very thankful for the development of the study due to the density of the graphs obtained while Cartesian product may be more complex for domination but provides more valuable results.

Most of the capabilities and knowledge have been put in practice during the realization of the work. On the other hand, this has helped us realizing of the importance and the impact of theoretical investigation in real implementation. Hence, while doing the investigation and research on new formulae, we have been able to test our thinking abilities in order to facilitate methodologies previously used.

6.2 Objectives

Our main aim was to work on research studies so that we could realize the amount of effort needed in order to bring new results to the current existing theory. By doing so, we wanted to achieve good results which can bring value to future implementations and research. As we could provide many bounds and formulae, we can say that the objectives were achieved successfully. The fact of providing new ideas and research to scientific world has motivated us to keep working on this field. In this way, we will be able to take part in the evolution of technologies, even if it is with a little contribution. We can say that a new goal has raised form this study.

6.3 Organization

Developing a project which is not being guided by the specifications of practices, also has supposed a challenge to overcome. This part has implied acquiring an own methodology of work and to strengthen the capability of developing new ideas. For doing so, it is very important be constant in the work and to keep informed about the topic. Only by investigation and continuity, valuable results can be achieved.

6.4 Open problems

Some closed formulae for $\gamma_r(G \circ H)$, obtained in Section 4.1.3, were derived under the assumption that $\gamma_t(G) = \frac{1}{2}\max\{\gamma_r(G), 2\rho(G)\}$ or $\gamma_{2,t}(G) = \max\{\gamma_r(G), 2\rho(G)\}$ or $\gamma_r(G) = 2\gamma(G)$. This suggests the following open problems.

Problem 1. *Characterize the graphs with* $\gamma_r(G) = 2\gamma(G)$.

Problem 2. *Characterize the graphs with* $\gamma_r(G) = 2\gamma_t(G)$.

Problem 3. *Characterize the graphs with* $\gamma_t(G) = \rho(G)$.

Problem 4. *Characterize the graphs with* $\gamma_r(G) = \gamma_{2,t}(G)$.

Problem 5. *Characterize the graphs with* $\gamma_{2,t}(G) = 2\rho(G)$.

Notice that $\gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G) \leq 2\gamma_t(G)$. Hence, $\gamma_r(G) = 2\gamma(G)$ if and only if $\gamma_r(G) = \gamma_R(G)$ and G is a Roman graph. Furthermore, $\gamma_r(G) = 2\gamma_t(G)$ if and only if all equalities hold true in the previous domination chain. Therefore, the starting point to solve Problems 1 and 2 is a deep investigation of Roman graphs.

Glossary

$\mathcal{P}_3(G)$ Family of graphs defined for the aim of this work. Given a graph G, let $\mathcal{P}_3(G)$ be the family of ordered sets $S = \{x_1, x_2, x_3\} \subset V(G)$ such that $\langle S \rangle \cong P_3$, $\delta(x_1) \geq 2$, $\delta(x_2) = 2$ and $\delta(x_3) = 1$.

3-cube graph Hypercube Q_3.

chromatic number Vertex-colouring function of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ with the property that $f(u) = f(v)$ whenever $\{u, v\} \in E(G)$ is defined as $\chi(G) = min_{f \in \mathcal{F}(G)} |Im(f)|$ where $Im(f)$ denotes the image of f.

clique Subset of vertices of a graph such that every two distinct vertices in the clique are adjacent; that is, its induced subgraph is complete.

clique covering number The minimum number of cliques of a given graph G needed to cover the vertex set $V(G)$ is called the clique covering number of G and denoted by $\theta(G)$.

Cocktail-party graph The Cocktail-party graph of order n is the graph consting of two rows of paired nodes in which all nodes except the paired nodes are connected.

comb Family of graphs defined for the aim of this work. Family of trees T_n, such that taking a path P_k of length $k = \lceil \frac{n}{3} \rceil$, with vertices $v_1, \ldots, v_k$, and attach a path P_3 to each vertex $v_1, \ldots, v_{k-1}$, by identifying each v_i with a leaf of its corresponding copy of P_3. Finally, we attach a path of length $r = n - 3\lceil \frac{n}{3} \rceil + 2$ to v_k.

complement Given the graph $G = (V, E)$, the complement of this graph is defined as the graph, $(G)^c$, that is constructed on the same set of vertices, so that two vertices are adjacent in $(G)^c$ if and only if they are not adjacent in G.

corona product Product of graphs obtained by creating a copy of the second factor for each of the nodes of the first one and joining oll the vertices of the copy to the corresponding vertex of the first graph.

cylinder graph Graph that is obtained by the Cartesian product of a cycle graph and a path graph of order n and m respectively: $C_n \Box P_m$.

density Amount of existing edges of a graph against the possible edges.

diameter Longest path among all shortest paths between all pair of nodes in a graph. The density of a graph increases as more adjacencies it has.

doubly connected domination number For a given connected graph $G = (V, E)$, a set $D \subseteq V(G)$ is a doubly connected dominating set if it is dominating and both $\langle D \rangle$ and $\langle V(G) - D \rangle$ are connected. The cardinality of the minimum doubly connected dominating set in G is the doubly connected domination number.

embedded subtrees of T Sequence of all embedded subtrees of T, of order greater than or equal to three: $T_0, T_1, \ldots, T_k$. $T_0 = T$ and T_i is the subtree of T_{i-1} induced by the removal of some suport vertices and leaves from T_{i-1} according to Definition 21.

family $\mathcal{O}_4(G)$ Family of graphs defined for the aim of this work. For any G such that Family $\mathcal{P}_4(G) \neq \emptyset$ we define the family $\mathcal{O}_4(G)$ of graphs G^* constructed from G as follows. Let $S \in \mathcal{P}_4(G)$ such that $\langle S \rangle = P_4 = (x_1, x_2, x_3, x_4)$, $X = \{x_1x_2, x_2x_3, x_3x_4\}$ and $Y = \{ab : a \in N(x_1) \setminus \{x_2\} \cup b \in N(x_4) \setminus \{x_3\}\}$. The vertex set of G^* is $V(G^*) = V(G) \setminus S$ and the edge set is $E(G^*) = (E(G) \setminus X) \cup Y$.

family $\mathcal{P}_4(G)$ Family of ordered sets defined for the aim of this work. Given a graph G, let $\mathcal{P}_4(G)$ be the family of ordered sets $S = \{x_1, x_2, x_3, x_4\} \subset V(G)$ such that $\langle S \rangle \cong P_4$, $\delta(x_1) \geq 2$, $\delta(x_2) = \delta(x_3) = 2$ and $\delta(x_4) \geq 2$.

family $\mathcal{G}$ Family of graphs defined for the aim of this paper. A graph $G_{r,s} = (V, E)$ belongs to $\mathcal{G}$ if and only if there exit two positive integers r, s such that $V = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s\}$ and $E = \{x_1y_i : 1 \leq i \leq r\} \cup \{x_1z_i : 1 \leq i \leq s\} \cup \{x_2y_i : 1 \leq i \leq r\} \cup \{x_3z_i : 1 \leq i \leq s\} \cup \{x_2x_3\}$. Figure 4.5 shows the graph $G_{4,4}$.

family $\mathcal{H}_k$ Family of graphs defined for the aim of this work. A graph $G = (V, E)$ belongs to family $\mathcal{H}_k$ if and only if it is constructed from a cycle C_k and k empty graphs $N_{s_1}, \ldots, N_{s_k}$ of order $s_1, \ldots, s_k$, respectively, and joining by an edge each vertex from N_{s_i} with the vertices v_i and v_{i+1} of C_k. Here we are assuming that v_i is adjacent to v_{i+1} in C_k, where the subscripts are taken module k. The graphs in this family of graphs have the property that $\gamma_r(G) = \gamma_{2,t}(G)$.

grid graph Graph that is obtained by the Cartesian product of two path graphs of order n and m respectively: $P_n \square P_m$.

Hamiltonian cycle Cycle that passes through each node in a Hamiltonian graph exactly once. An example of a Hamiltonian cycle is shown in Figure 3.1.

Hamiltonian graph Graph such that contains a cycle that passes through each node exactly once. An example of a Hamiltonian graph is shown in Figure 3.1.

Hamming graph The Hamming graph, denoted by $H_{k,t}$, is the Cartesian product of k copies of the complete graph K_t.

hypercube Hypercubes, Q_t are the family of Hamming graphs, denoted by $H_{t,2}$, generated by the Cartesian product of K_2.

independent edge set Subset of edges such that no two edges in the subset share a vertex.

matching number The matching number $\alpha'(G)$ of graph G, sometimes known as the edge independence number, is the cardinality of a maximum independent edge set.

maximum twin set Definition provided for the aim of this work. Let $\mathcal{D}(G)$ be the set of all $\gamma(G)$-sets. For every $S \in \mathcal{D}(G)$ we define

$$T(S) = \{v \in V(G) \setminus S : \; N[v] = N[s] \text{ for some } s \in S\}.$$

Finally, we define

$$\tau(G) = \max\{|T(S)| : \; S \in \mathcal{D}(G)\}$$

as the maximum twin set.

non-universal vertex An universal vertex v is a vertex of maximum degree, $\delta(v) = n - 1$ in a graph of order n. A non-universal vertex v' is then a vertex of degree $\delta(v') < n - 1$.

planar graph Graph that can be drawn in such a way that no edges cross each other.

property $\mathcal{P}'$ Let H be a graph. A vertex $a \in V(H)$ satisfies Property $\mathcal{P}'$ if $\{a, b\}$ is a dominating set of H, for every $b \in V(H) \setminus N[a]$. In other words, $a \in V(H)$ satisfies $\mathcal{P}'$ if the subgraph induced by $V(H) \setminus N[a]$ is a clique.

property $\mathcal{P}$ Property defined in Section 4 used for charectirising noncomplete graphs with two nodes $\{a, b\} \subseteq V(G)$ satisfying the following conditions:

- $\{a, b\}$ is a dominating set.
- If $x \in V(G) \setminus N[a]$, then $\{x, a\}$ is a dominating set.
- If $x \in V(G) \setminus N[b]$, then $\{x, b\}$ is a dominating set.
- If $x \in N(a) \cap N(b)$, then $\{x, a\}$ is a dominating set or $\{x, b\}$ is a dominating set

.

rainbow domination number Given a graph G, we have a set of k colors and assign an arbitrary subset of these colors to each vertex of G. If a vertex which is assigned an empty set, then the union of color set of its neighbors must be k colors. This assignment is called the k-rainbow dominating function of G. The weight of the function is the sum of numbers of assigned colors over all vertices of G. The minimum weight of this k-rainbow dominating function is defined as the k-rainbow domination number of G..

self-complementary graph Graph G whose complement is isomorphic to the same graph $(G)^c \cong G$.

shortest path Path of minimum cardinality between two given vertices in a graph.

super domination number The open neighbourhood of a vertex v of a graph G is the set $N(v)$ consisting of all vertices adjacent to v in G. For $D \subseteq V(G)$, we define $\overline{D} = V(G) \setminus D$. A set $D \subseteq V(G)$ is called a super dominating set of G if for every vertex $u \in \overline{D}$, there exists $v \in D$ such that $N(v) \cap \overline{D} = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating sets in G.

torus graph Graph that is obtained by the Cartesian product of two cycle graphs of order n and m respectively: $C_n \square C_m$.

true twin Vertex u of graph G is a true twin of $v \in V(G)$ if $N[u] = N[v]$.

Acronyms

C_n Cycle graph of order n.

$K_{1,n-1}$ Star graph of order n.

K_n Complte graph of order n.

$K_{r,s}$ Complte bipartite graph of order $n = r + s$.

N_n Empty graph of order n.

P_n Path graph of order n.

T_{3k} Comb graph of order $n = 3k$ defined for the aim of this work according to Definition 19.

T_n Comb graph of order n defined for the aim of this work according to Definition 19.

DF Dominating Function.

RDF Roman Dominating Function.

SDF Secure Dominating Function.

WRDF Weak Roman Dominating Function.

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Annex 1

Proof of Theorem 37

To prove Theorem 37 wee need the following lemma.

Lemma 65. *Let G and H be nontrivial connected graphs. If $\gamma(H) \geq 4$, then there exists a $\gamma_r(G \circ H)$-function f such that $\sum_{u' \in N(u)} f(H_{u'}) \geq 2$, for every $u \in V(G)$.*

Proof. Let $u, u' \in V(G)$ such that $u' \in N(u)$ and $v' \in V(H)$. First, suppose that $\sum_{z \in N(u)} f(H_z) = f(u', v') = 1$. If $f(H_u) < \gamma(H) - 1$, then there exists $v \in V(H)$ such that $\sum_{h \in N[v]} f(u, h) = 0$, so that the movement of the legion from (u', v') to (u, v) produces unprotected vertices, which is a contradiction. Hence, $f(H_u) \geq \gamma(H) - 1 \geq 3$ and we can construct a $\gamma_r(G \circ H)$-function f_1 from f as follows. For some (u, v) such that $f(u, v) \geq 1$ we set $f_1(u, v) = f(u, v) - 1$, for some $v'' \neq v'$ we set $f_1(u', v'') = 1$ and $f_1(x, y) = f(x, y)$ for every $(x, y) \in V(G \circ H) \setminus \{(u, v), (u', v'')\}$. Hence, $\sum_{z \in N(u)} f_1(H_z) = f_1(u', v') + f_1(u', v'') = 2$.

Now, if $\sum_{z \in N(u)} f(H_z) = 0$, then we proceed as above to construct a $\gamma_r(G \circ H)$-function f_1 from f by the movement of two legions from H_u to (u', v'). In this case, $\sum_{z \in N(u)} f_1(H_z) = f_1(u', v') = 2$.

For each $u \in V(G)$ such that $\sum_{u' \in N(u)} f(H_{u'}) \leq 1$ we can repeat the procedure above until finally obtaining a $\gamma_r(G \circ H)$-function satisfying the result. $\square$

Proof of Theorem 37. Let $S \in \mathcal{P}_4(G)$ such that $\langle S \rangle \cong P_4 = (x_1, x_2, x_3, x_4)$. We will first show that $\gamma_r(G \circ H) \leq \gamma_r(G^* \circ H) + 4$. Let f be a $\gamma_r(G^* \circ H)$-function and define α_1 and α_4 as follows:

$$\alpha_1 = \sum_{x \in N(x_1) \setminus \{x_2\}} f(H_x) \quad \text{and} \quad \alpha_4 = \sum_{x \in N(x_4) \setminus \{x_3\}} f(H_x).$$

We will construct a WRDF f_1 on $G \circ H$ from f such that $w(f_1) \leq w(f) + 4$. For each vertex $(u, v) \in V(G^* \circ H)$ we set $f_1(u, v) = f(u, v)$ and now we will describe the following six cases for the vertices $(u, v) \in S \times V(H)$, where symmetric cases are omitted. In all these cases we fix $y \in V(H)$.

1. $\alpha_1 \geq 2$ and $\alpha_4 \geq 2$. We set $f_1(x_1, y) = f_1(x_4, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_1, y), (x_4, y)\}$.
2. $\alpha_1 \geq 2$ and $\alpha_4 = 1$. We set $f_1(x_1, y) = f_1(x_3, y) = 1$, $f_1(x_4, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_1, y), (x_3, y), (x_4, y)\}$.
3. $\alpha_1 \geq 2$ and $\alpha_4 = 0$. We set $f_1(x_3, y) = f_1(x_4, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_3, y), (x_4, y)\}$.
4. $\alpha_1 = 1$ and $\alpha_4 = 1$. We set $f_1(x_1, y) = f_1(x_2, y) = f_1(x_3, y) = f_1(x_4, y) = 1$ and $f_1(u, v) = 0$ for every $v \neq y$ and $u \notin \{x_1, x_2, x_3, x_4\}$.
5. $\alpha_1 = 1$ and $\alpha_4 = 0$. We set $f_1(x_2, y) = f_1(x_4, y) = 1$, $f_1(x_3, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_2, y), (x_3, y), (x_4, y)\}$.
6. $\alpha_1 = 0$ and $\alpha_4 = 0$. We set $f_1(x_2, y) = f_1(x_3, y) = 2$ and $f_1(u, v) = 0$ for every $(u, v) \notin \{(x_2, y), (x_3, y)\}$.

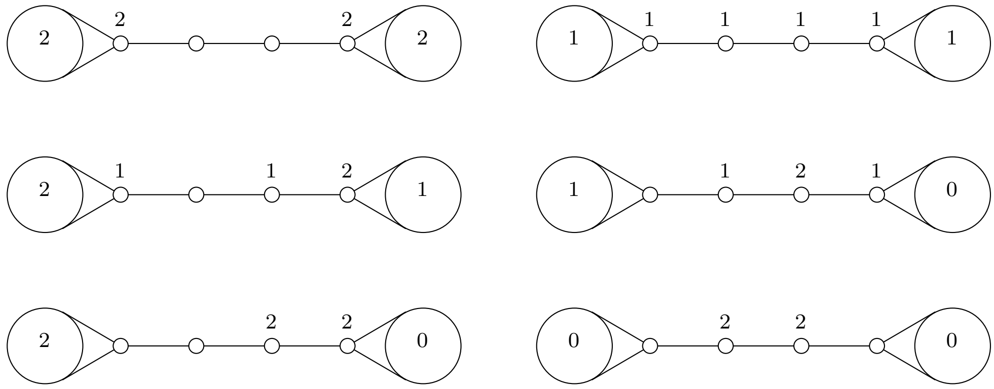

Figure 1.1: Representation of the placement of entities for each of the cases for vertices $(u, v) \in S \times V(H)$.

A simple case analysis shows that the vertices of $G \circ H$ are protected by the assignment of legions produced by f_1. Therefore,

$$\gamma_r(G \circ H) \leq w(f_1) \leq w(f) + 4 = \gamma_r(G^* \circ H) + 4. \tag{1.1}$$

Now we will show that the equality holds. Let g be a $\gamma_r(G \circ H)$-function satisfying Lemma 65. We will construct a WRDF g_1 on $G^* \circ H$ from the function g such that $w(g_1) \leq w(g) - 4$. We also need to define β_1 and β_4 as follows.

$$\beta_1 = \bigcup_{x \in N(x_1) \setminus \{x_2\}} V(H_x) \quad \text{and} \quad \beta_4 = \bigcup_{x \in N(x_4) \setminus \{x_3\}} V(H_x).$$

We define g_1 according to the following six cases:

1'. $g(\beta_1) \geq 2$ and $g(\beta_4) \geq 2$. In this case, we set $g_1(x,y) = g(x,y)$ for every $(x,y) \in V(G^* \circ H)$.

2'. $g(\beta_1) \geq 2$ and $g(\beta_4) = 1$. Depending on $g(H_1)$ we will consider the following two cases:

2'.1 $g(H_1) \leq 1$. Since $g(H_1) \leq g(\beta_4)$, we can set $g_1(x,y) = g(x,y)$ for for every $(x,y) \in V(G^* \circ H)$.

2'.2 $g(H_1) \geq 2$. We will show that $g(S \times V(H)) \geq 5$. To see this, we will try to place four legions in $S \times V(H)$ as shown in Figure 1.2, where $0 \leq a, b \leq 2$. Since in all these cases we have a contradiction with Lemma 65, we can conclude that $g(S \times V(H)) \geq 5$. Hence, we place the legions in the following way: for some $(x_0, y_0) \in \beta_4$ we set $g_1(x_0, y_0) = g(x_0, y_0) + 1$ and $g_1(x,y) = g(x,y)$ for every $(x,y) \in (G^* \circ H) \setminus \{(x_0, y_0)\}$.

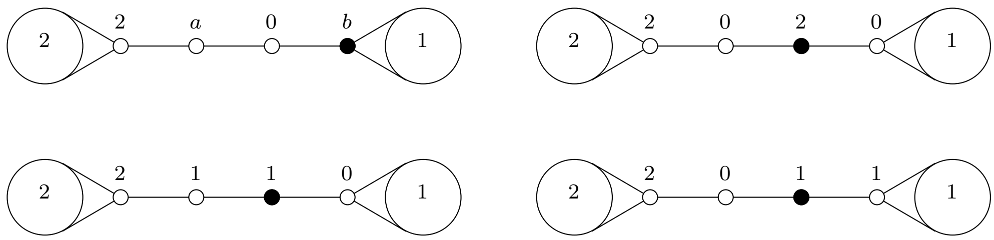

Figure 1.2: *Proof of Theorem 37*: Scheme corresponding to Case 2'.2.

3'. $g(\beta_1) \geq 2$ and $g(\beta_4) = 0$. In this case, we consider the following three cases depending on the value of $g(H_1)$:

3'.1 $g(H_1) = 0$. In this case $g(H_1) = g(\beta_4)$ so we set $g_1(x,y) = g(x,y)$ for every $(x,y) \in V(G^* \circ H)$.

3'.2 $g(H_1) = 1$. We will show that $g(S \times V(H)) \geq 5$. To see this, we will try to place four legions in $S \times V(H)$ as shown in Figure 1.3, where $2 \leq a + b \leq 3$ and $c + d = 1$. In both cases we have a contradiction with Lemma 65. Hence, $g(S \times V(H)) \geq 5$ and so

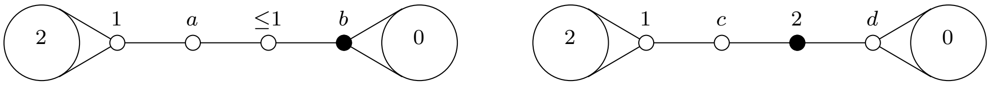

Figure 1.3: *Proof of Theorem 37*: Scheme corresponding to Case 3'.2.

we place the legions in the following way: for some $(x_0, y_0) \in \beta_4$ we set $g_1(x_0, y_0) = 1$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0)\}$.

3'.3 $g(H_1) \geq 2$. We will show that $g(S \times V(H)) \geq 6$. To see this, we will try to place five legions in $S \times V(H)$ as shown in Figure 1.4, where $2 \leq a + b \leq 3$ and $c + d = 1$. In both cases we have a contradiction with Lemma 65. Hence, $g(S \times V(H)) \geq 6$ and so we place the legions in the following way: for some $(x_0, y_0) \in \beta_4$ we set $g_1(x_0, y_0) = 2$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0)\}$.

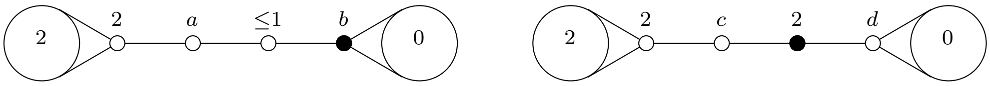

Figure 1.4: *Proof of Theorem 37*: Scheme corresponding to Case 3'.3.

4' $g(\beta_1) = g(\beta_4) = 1$. In this case, we consider the following three cases depending on the value of $g(H_1)$ and $g(H_4)$:

4'.1 $g(H_1) \leq 1$ and $g(H_4) \leq 1$. In this case $g(H_1) \leq g(\beta_4)$ and $g(H_4) \leq g(\beta_1)$, so we set $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H)$.

4'.2 $g(H_1) \geq 2$ and $g(H_4) \leq 1$ (this case is symmetric to $g(H_1) \leq 1$ and $g(H_4) \geq 2$). We will show that $g(S \times V(H)) \geq 5$. To see this, we will try to place four legions in $S \times V(H)$ as shown in Figure 1.5, where $a + b = 1$. In all cases we have a contradiction with Lemma 65. Hence, $g(S \times V(H)) \geq 5$ and so we define g_1 as follows: for some $(x_0, y_0) \in \beta_4$ we set $g_1(x_0, y_0) = g(x_0, y_0) + 1$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0)\}$.

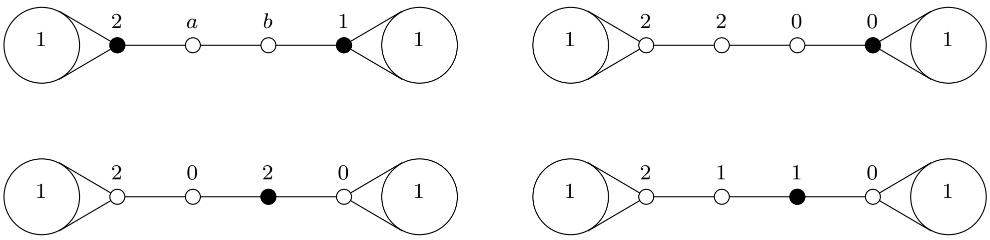

Figure 1.5: *Proof of Theorem 37*: Scheme corresponding to Case 4'.2.

4'.3 $g(H_1) \geq 2$ and $g(H_4) \geq 2$.

We will show that $g(S \times V(H)) \geq 6$. To see this, we will try to place five legions in $S \times V(H)$ as shown in Figure 1.6, where $a + b = 1$. In this case we have a contradiction with Lemma 65. Hence, $g(S \times V(H)) \geq 6$ and so we place the legions in the following way: for some $(x_0, y_0) \in \beta_1$ we set $g_1(x_0, y_0) = g(x_0, y_0) + 1$, for some $(x'_0, y'_0) \in \beta_4$ we set $g_1(x'_0, y'_0) = g(x'_0, y'_0) + 1$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x'_0, y'_0)\}$.

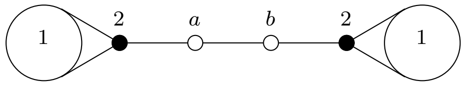

Figure 1.6: *Proof of Theorem 37*: Scheme corresponding to Case 4'.3.

5' $g(\beta_1) = 1$ and $g(\beta_4) = 0$. Notice that if $g(H_2) = 0$ or $g(H_3) \leq 1$, then we have a contradiction with Lemma 65, so that $g(H_2) \geq 1$ and $g(H_3) \geq 2$. We differentiate two cases according to the value of $g(H_2)$:

5'.1 $g(H_2) = 1$. By Lemma 65, we have that $g(H_4) \geq 1$. Thus, we place the legions in the following way: for some $(x_0, y_0) \in \beta_1$ we set $g_1(x_0, y_0) = \min\{2, g(H_4) - 1\}$, for some $(x'_0, y'_0) \in \beta_4$ we set $g_1(x'_0, y'_0) = \min\{2, g(H_1)\}$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x'_0, y'_0)\}$.

5'.2 $g(H_2) \geq 2$. In this case, we place the legions in the following way: for some $(x_0, y_0) \in \beta_1$ we set $g_1(x_0, y_0) = \min\{2, g(H_4)\}$, for some $(x'_0, y'_0) \in \beta_4$ we set $g_1(x'_0, y'_0) = \min\{2, g(H_1)\}$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x'_0, y'_0)\}$.

6' $g(\beta_1) = g(\beta_4) = 0$. Notice that if $g(H_2) \leq 1$ or $g(H_3) \leq 1$, then we have a contradiction with Lemma 65, so that $g(H_2) \geq 2$ and $g(H_3) \geq 2$. We place the legions in the following way: for some $(x_0, y_0) \in \beta_1$ we set $g_1(x_0, y_0) = \min\{2, g(H_4)\}$, for some $(x'_0, y'_0) \in \beta_4$ we set $g_1(x'_0, y'_0) = \min\{2, g(H_1)\}$ and $g_1(x, y) = g(x, y)$ for every $(x, y) \in V(G^* \circ H) \setminus \{(x_0, y_0), (x'_0, y'_0)\}$.

A simple case analysis shows that the vertices of $G^* \circ H$ are protected by the assignment of legions produced by g_1. Therefore,

$$\gamma_r(G^* \circ H) \leq w(g_1) \leq w(g) - 4 \leq \gamma_r(G \circ H) - 4. \tag{1.2}$$

Finally, by (1.1) and (1.2) we can conclude that $\gamma_r(G \circ H) = \gamma_r(G^* \circ H) + 4$, as claimed. $\square$