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What is an attractor?

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I would like to dedicate this paper to Beatriz, my mum, and all the people who encouraged me to do this master and believed in me. Thank you.

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Abstract

In this paper we will mainly discuss about dynamical systems and the notion of attractor, an important concept which allow us to obtain a lot of information about the behaviour of dynamical systems. The first chapters are mainly dedicated to give the definition of both concepts, as well as some of their most important properties. We will see there are two types of dynamical systems depending on the time set we consider.

Afterwards, we will show some examples of attractors focusing on fixed points and two important examples of attractor in the literature: the Feigenabum attractor, which appears on the study of the quadratic family, in both real and complex spaces, and the Lorenz attractor, which appears on the study of a three-dimensional linear system. In both cases, we will do a MATLAB simulation of the attractors in order to visually see how they behave and the various shapes they are able to adopt.

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Chapter 1

Introduction

1.1 Context and justification of the Work

Until the end of the 19th century, all mathematical studies were mainly concerned with describing the quantitative properties of the problem in question, something that allowed us to achieve great mathematical advances in many areas, but which did not allow major breakthroughs in the n-Body Problem (the problem that tries to understand the performance of the planetary system given n planets). In order to carry out a more exhaustive study of the 3-body problem (Sun-Earth-Moon), given that it was impossible to obtain explicit formulas for its equations, Poincaré used a totally innovative method based on the study of the system as a whole, giving priority to the qualitative properties of the system instead of the quantitative ones. This new method defined the basis of the Dynamical Systems Theory.

As one could expect, since the beginning the utility of the dynamical systems has been deeply rooted to Celestial Mechanics and Analytical Mechanics, but little by little it has been making its way into different areas of great importance. Within the area of mathematics, it is naturally related to practically all its branches (analysis: theory of functions, functional analysis, measure theory; numbers theory; geometry: differential geometry, algebraic geometry, etc.) but, also it is related to areas more distant from mathematics, such as biology or economics.

In this context, which seems so favourable for the application of dynamic systems, there are also some problems that are important to highlight. A basic problem is to calculate what can be called "the skeleton of the Dynamical System", i.e. the geometrical objects that guide the dynamics of the system. They are invariant objects under the action of the system which allow us to obtain a lot of information about the system by simply making a study in a neighborhood close to them. In this dissertation, the geometric objects in which we will focus will be fixed points and periodic points which act like an attractor. The aim of this work is to show some of the attractors that can be found in both linear and quadratic functions that allow us to deduce from the information gathered some general properties of the system.

If we put together all the information obtained from these objects, we can find a path that lead us to the general solutions to the problem. But, if the path is complicated, invariant objects may appear that are not varieties, such as the so-called "strange attractors".

Another problem we may encounter in the study of these dynamical systems is how to prove the existence of certain solutions. Using topological and geometrical methods, to find a solution can be quite exhausting and often all the work done does not mean significant progress. To solve this problem, well-known solutions of simpler systems (reductions of the original system) may be tackled by analytical methods which can provide us with really interesting properties of the original system. Between these two approaches there is a very large land where rigorous numerical methods are also indispensable.

The computation of these geometrical objects presents significant challenges even for the most refined numerical methods, and it is one of the most important problems to be tackled nowadays.

1.2 Aims of the Work

1. To explain the utility of the concept of attractor in the field of dynamical systems.

2. To state the different definitions of attractor that have appeared since its beginnings, classifying them from the most restrictive to the least restrictive.

3. To state the formal definition of attractor that we have chosen to use in this dissertation. Important concepts related to the concept of attractor are also given for a better understanding.

4. To show some of the best-known examples of attractors, with special emphasis on fixed and periodic points in both discrete and continuous dynamical systems.

5. To present the Feigenabum Attractor and the Lorenz Attractor, two important attractors of special relevance at present.

1.3 Approach and method followed

At first, I started reading articles about dynamical systems that allowed me to get introduced to the field. Once I had an idea about how dynamical systems worked and the role that attractors played on them, I started to read more specific articles dealing with the concept of attractors, from the easiest to the most elaborated ones. Among them, the main articles on which I have based part of my work have been Milnor [7] for Chapters 3-5, Lyubich [6] for Chapter 5 and Tucker [11] for Chapter 6.

Once I had read all the articles, I made a video call with Antonio Garijo (my tutor) to discuss the work and present the idea I had in mind. Once we have decided about the best structure of this dissertation, I started working on it. Firstly I wrote a first version of the work that I sent to Antonio, and after he gave me his approval, we worked hand-in-hand to improve each of the chapters until we obtained the final version of the dissertation.

In the performance of this work, I have mainly focused on the mathematical study of the attractors, although I have also used my computer skills to simulate some of the attractors described in theory for a better understanding of the problem.

1.4 Planning of the Work

This work has been carried out since the beginning of the academic year (end of September) until now. From September to December I was reading all the readings and articles related to attractors to get into the subject and make up my mind about the path I wanted to take in this dissertation. Once I had a good idea of the subject, at the end of December (before Christmas) I did a video call with Antonio Garijo and we discussed what was the better path I had to take to do this project. In this video call everything was very clear, so shortly after, I started to write all the pages of the work.

From January to June I have been in constant contact con Antonio, firstly (from January to March) I did a first version of the dissertation (it was about 30 pages) where I had already wrote about dynamical systems, the definition of attractor, fixed points and something about the Feigenbaum attractor. Then, I sent it by email to Antonio so he could tell me about what other things I could add to the dissertation to make it better. Antonio told me to add some more examples in each chapter, as well as some simulations to show better what the behaviour of the attractor was, then I had also the idea to include the Lorenz attractor since it is an important example of attractor in the literature. For May, I had written the second version of the work. I sent him another version of the project with 50 pages which was nearer to the final version of the work, but still needed some complementary explanations to some parts of the theory that were still very confusing.

From May to June I have been exchanging a few versions of the work with Antonio, mainly with improvements in the theory that allowed the reader to have an easier reading as well as more elaborate explanations which provides a better understanding of the concepts.

1.5 Brief summary of products obtained

1. The definition of dynamical systems.

2. The definition of attractor.

3. Examples of attracting points in both discrete and continuous dynamical systems.

4. A detailed study of the Feigenbaum attractor, in the real and complex space. Some simulations of the attractor are also given.

5. A detailed proof that demonstrate the Lorenz attractor indeed exists. Some simulations of the attractor are also given together with some of its most important properties.

1.6 Brief description of the others chapters of the memory

This dissertation is divided into six chapters, the first one (the current one as an introduction) and five more which are organized in the following way.

The second chapter consists of four sections. The first one gives a brief historical introduction to the concept of dynamical systems and the second section gives the formal definition of a dynamical system and a detailed description of how it works. After reading these two chapters it should be clear what a dynamical system is and the role of attractors in them. In the third section we present the concepts of “Lyapunov stable” and “asymptotically stable” that we will use in future chapters (these concepts have been closely related to the concept of attractor since its beginnings) and finally, in the fourth one, we present the Cantor set, a set that appears more than once during the work and that is necessary to know in order to understand the structure of some attractors.

The third chapter introduces the concept of an attractor. It is divided in two sections, in the first one our aim is to present different definitions of attractor since their emergence, while in the second one we introduce the formal definition we will use throughtout the work. At the end of this chapter it should be clear what is an attractor and what are their most important properties.

In the fourth chapter we present some the easiest examples of attractor we can find in the literature, the fixed points. This chapter is divided in two sections, in the first one we define what are fixed and periodic points in discrete continuous systems, together with six examples of attractors. In the second section, we define what is a critical point and the different types of critical points we can find (some will be attractor, some will not) as well as we show seven examples for a better understanding of the subject.

In chapters five and six, we present two important attractors in the literature. In chapter five, the Feigenbaum attractor (which is an attractor given in a discrete dynamical system), while in chapter six we have the Lorenz attractor (an attractor given in a continuous dynamical system). In the first one our aim is to study the Feigenbaum attractor in both real and complex spaces while in the last chapter our aim is to show some simulations about the Lorenz attractor, to explain some of its more important properties and finally, to proof it indeed exists (even though this attractor was found by Lorenz in the beginning of the 20th century, nobody was able to demonstrate that it indeed exists until a few years ago).

Chapter 2

Introduction to Dynamical Systems

2.1 History

For a better understanding of attractors, which is their role and how important are they, firstly, we need to take a look at the origins of the dynamical systems (see [8] for more historical information about them).

Unexpectedly, its origins are based in a contest that was arranged in 1889 in commemoration to the 60th anniversary of Oscar II, King of Sweden. Mathematicians from all Europe were invited to write an original paper addressing one of four questions put by Weierstrass, who was in that moment a central figure in mathematics. One of these questions concerned celestial mechanics, Weierstrass' purpose was to find a person who was able to demonstrate the stability of the solar system proving the convergence of the series derived from the solutions of the equations that governed celestial mechanics, something that he was not able to do, even though he firmly believed in it.

Soon after, Poincaré presented to this contest a wide and complex paper proving the long-awaited stability of the solar system using a new (and unique) geometrical perspective that allowed him to visualize a global picture of the dynamics, instead of using series as Weierstrass expected. The judges were so impressed by his work that Poincaré was claimed the winner of the contest. Unfortunately, shortly after it was proven that he was wrong.

Poincaré needed two years to revise his proof and submit a new paper, a long and exhausting task that led to the emergency of the theory of dynamical systems. In this revised paper, named *"Sur le problem des trois corps et les equations de la dynamique"*, Poincaré defined some imaginary portraits called "phase space" that served as the backdrop for the geometrical shapes and flows that represents the totality of the solutions of any differential equation. These geometrical portraits revealed instability and a perplexing dynamical domain of wondrous complexity which shattered any theory of stability of the solar system. Unknowingly, this investigations knocked softly what we know nowadays as dynamical chaos.

Poincaré's explorations of this curious parallelism between chaos and order, in which a clock's pendulum or a solar system governed by the laws of Newtonian mechanics can display such complicated dynamics, provoked a host of new questions (most of them remain open nowadays).

Over the next 70 years, only a handful of mathematicians continued along the course that Poincaré had set in the geometrical study of dynamical systems. Among them was George Bikhoff, who proved in 1913 Poincaré's last geometric theorem (a special case of the Three-Body Problem) and Vladimir Arnol'd, who finally managed to solve Poincaré's problem in 1963 with astonishing results. It was proven that under certain conditions, the series used to describe motions in the three-body problem did converge, but under another set of conditions they did not. Thus, the conclusion obtained was that depending on the initial conditions, motion in a system of three or more bodies is sometimes regular and sometimes chaotic, defining what we call nowadays chaotic dynamics.

Apart from this, no great discoveries appeared on this field until the concept of attractor was discovered. In this complex part of the dynamics, where "phace spaces" show the rates of change of each of the coordinates through a direction of travel for each particle at any instant, we can see "trajectories" and "orbits" which do not follow a defined structure but wander around some well-defined movement (see figure 2.1). When this happens, the system is said to be attracted to some "kind" of motion, i.e. an attractor.

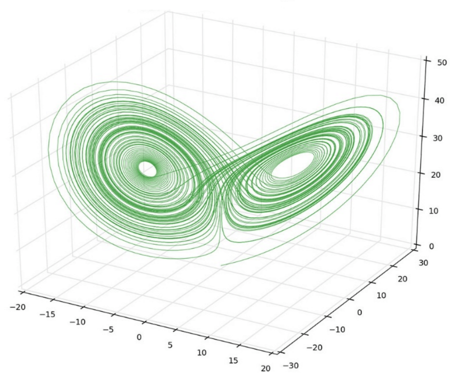

Figure 2.1: Phase space of the Lorentz attractor.

2.2 What is a dynamical system?

Roughly speaking, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. Our aim in this section is to give a detailed explanation of what they are and how they work.

Definition 2.1 *It is called a dynamical system to the set (S, T, ϕ_t) where S is the set of states (i.e. set of variables), T is the time set and $\{\phi_t\}$ is the set of maps $\phi_t : S \to S$ that satisfy:*

1. $\phi_0 = id$.

2. $\phi_{t_1+t_2} = \phi_{t_2} \circ \phi_{t_1} = \phi_{t_1} \circ \phi_{t_2}\ \forall t_1, t_2 \in T$.

In the case we set $T = \mathbb{N}$ or $\mathbb{Z}$, it is called a discrete dynamical system. Otherwise (if we set $T = \mathbb{R}$ for example) it is called a continuous dynamical system.

A dynamical system can be expressed as an iterative method, such as the one-dimensional Newton's method $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $i \in \mathbb{N}$. But, also, dynamical systems can be defined using a continuous map ϕ_t as in the following examples.

Example 2.1 *We consider the time set $T = \mathbb{R}$, the set of states $S = \mathbb{R}$ and the map $\phi_t(x) = e^{at}x$ for $a \in \mathbb{R}$. We prove it is indeed a dynamical system, to do so, we check the two conditions that must satisfy ϕ_t.*

1. $\phi_0(x) = e^0 x = x$.

2. $\phi_{t_1+t_2} = e^{a(t_1+t_2)}x = e^{at_1 at_2}x = \phi_{t_2} \circ \phi_{t_1}$ ✓

Example 2.2 *We consider the time set $T = \mathbb{Z}$, the set of states $S = \mathbb{R}$ and the map $\phi_t(x) = x^{2t}$. As before, we prove it is a dynamical system as follows,*

1. $\phi_0(x) = x^{2^0} = x^1 = x$.

2. $\phi_{t_1+t_2} = x^{2^{t_1+t_2}} = x^{2^{t_1}2^{t_2}} = (x^{2t_1})^{2t_2} = \phi_{t_2} \circ \phi_{t_1}$ ✓

From Definition 2.1 we deduce the variables of a dynamical system are constantly being modified to produce changes over time. The way these changes occur defines the behaviour of the system.

In this dissertation, we will focus in dynamical systems where this behaviour involves some kind of attraction to a particular point or set.

Dynamic systems can also be used to analyse how small changes that occur only in one part of the system can affect the behaviour of the whole system. In particular, we will see how small changes in the initial conditions of a system can derive in a completely different behaviour of the system.

2.2.1 Discrete Dynamical Systems

A discrete dynamical system is given by an equation of the form

$$x_{k+1} = f(x_k),\ k = 0, 1, 2, ... \tag{2.1}$$

where f represents a map $f : M \to M$ where M can be considered as $M = \mathbb{C}^n$, $M = \mathbb{R}^n$, a Riemann Surface or even a topological space (in particular, for our examples we will mainly consider the first two cases). This set receives the name of the "phase space". Also, we will always assume f is a smooth map, it means, a function with continuous derivatives of all necessary orders (generally, $f \in C^1$).

The variables x_k that describe the system are called "state variables". They are grouped into a vector, known as the "state vector", which stores the complete information about the state of the system. Bearing this in mind, the phase space can be considered as the set of all possible state vectors of the system.

The equation of a dynamical system can be interpreted as follows. If the system adopts at an instant k a state described by a certain element x_k, then, at instant $k+1$ the state of the system will be x_{k+1}. Our smooth map f therefore represents the law of evolution of the dynamical system which transforms each state into the next state the system adopts.

If the system has as initial state x_0, then, the following states the system will go through over time corresponds to the sequence $\{(x_0, x_1, ..., x_n),\ n \in \mathbb{N}\}$ which is called the solution of the system given an initial condition x_0. Therefore, the solution of a dynamical system can be obtained recursively as follows:

$$\{x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), ...\}$$

In general, each term of the sequence is given by $x_k = f^k(x_0)$ where f^k represents the composition of f with itself k times. Without considering any initial condition, we can define the function $f^k(x)$ which represents the general solution to the system, also called, the flow of the discrete dynamical system. This function allow us to know the state of the dynamical system at any instant k given an initial condition x_0 just evaluating the function in x_0.

Once we know how these dynamical systems work, we can focus on studying their behavior. The easiest way to do it is to understand the nature of all the orbits of the map. In many cases, orbits can be quite complicated sets of points, even for linear mappings, however, there are some orbits which are especially simple and which play a central role on the study of the systems.

Definition 2.2 *The set of points* $\{x, f(x), f^2(x), ...\}$ *is called the forward orbit of* x *and is denoted by* $O^+(x)$*. If* f *is a homeomorphism (a function* f *one-to-one, onto and continuous with* $f^{-1}(x)$ *also continuous), we may define the full orbit of* x *denoted as* $O(x)$ *as the set of points* $\{f^n(x),\ n \in \mathbb{Z}\}$ *and the backward orbit of* x *denoted by* $O^-(x)$ *as the set of points* $\{x, f^{-1}(x), f^{-2}(x), ...\}$.

To see how an orbit works, it is easy to carry out the following experiment. Firstly, we put a number on the calculator (for example, 0.20) and then, we press the 10^x-key repeatedly. We will obtain the forward orbit of $x = 0.20$ given by

$$\{0.20, 10^{0.20}, 10^{10^{0.20}}, ...\}.$$

Actually, if we keep repeating it for several times, sometime we will obtain an error message from the calculator. This happens because the orbit tends to infinity (and then, the sequence diverge), but, if we take for example $f(x) = \sin x$ or $f(x) = \cos x$, we will see that in this case the error does not appear (this is because those sequences are convergent). This is just an example of different behaviours that a dynamical system can adopt.

2.2.2 Continuous Dynamical Systems

In this dissertation, we will only work with continuous dynamical systems which are given by a differential equation of the form

$$\dot{x} = f(t, x) \tag{2.2}$$

where the notation $\dot{x}$ represents the derivative of x by t defined as $\dot{x} = \frac{dx}{dt}$. As before, x defines a vector in $\mathbb{R}^n$ or $\mathbb{C}^n$ depending the case we consider and $t \in \mathbb{R}$, also f represents a smooth map $f : M \subset \mathbb{K}^{n+1} \to M \subset \mathbb{K}^n$ where $K = \mathbb{R}$ or $\mathbb{C}$ depending on the case we study. This set is called the "phase space" of the dynamical system and it describes the behaviour of the state variables $x_1, ..., x_n$. In particular, a point $(x_1(t_0), ..., x_n(t_0))$ at a particular moment t_0 is called a phase-point, and if we increase the value of t to obtain a solution $x(t)$ over a greater period of time (let $t \in I$) the point is supposed to move through the phase-space drawing a trajectory in the plane.

In continuous dynamical systems, the vector function $x(t_0)$ defined as $x(t_0) = (x_1(t_0), ..., x_n(t_0))$ represents a particular solution of Equation (2.2) at an instant t_0. In general, we can define the function $x(t)$ as the general solution of the system, also called, the flow of the continuous dynamical system, if it is defined on an interval $I \subset \mathbb{R}$ of t, the map $x : I \to \mathbb{R}^n$ or $\mathbb{C}^n$ is smooth (generally, $f \in C^1$) and of course satisfies Equation (2.2). This function allow us to know the state of a dynamical system at any instant t_0 just evaluating the function in this point.

Throughout the work, we will only consider differential equations in which the independent variable t does not occur explicitly, it means, of the form $\dot{x} = f(x)$ such that $x \in \mathbb{R}^n$ or $\mathbb{C}^n$. This kind of equations are called "autonomous" equations. In the following lines, we define what is an orbit for this type of differential equations.

Definition 2.3 *Reformulating the autonomous equation in components as* $\dot{x}_i = f_i(x)$. *Let* $O_i(x)$ *for* $i = 1, ..., n-1$ *be the solutions of the system*

$$\begin{aligned} \frac{dx_2}{dx_1} &= \frac{f_2(x)}{f_1(x)} \\ \vdots \quad & \quad \vdots \\ \frac{dx_n}{dx_1} &= \frac{f_n(x)}{f_1(x)} \end{aligned}$$

given $f_1(x) \neq 0$. *Then, the solutions* $O_i(x)$ *are called orbits.*

Orbits from the phase space will never intersect (this is a result from the existence and uniqueness theorem given in [13], page 3). In the case where $f_1(x) = 0$ but $f_i(x) \neq 0$, for $i \neq 1$, we can interchange the roles of these functions to be able to apply the previous definition.

2.3 Main concepts about Stability

Another important concept we will study is the stability of a dynamical system, a concept closely related to attractors since their emergence. The most important definitions of stability that we will use throughout the work are defined below.

Definition 2.4 *Let $f : M \to M$ be a fixed smooth map where M can be considered as $M = \mathbb{C}^n$, $M = \mathbb{R}^n$ or a smooth compact manifold. We say a closed subset $A \in M$ with $f(A) = A$ is Lyapunov stable if it has arbitrarily small neighborhoods U such that $f(U) \in U$.*

Definition 2.5 *Let $f : M \to M$ be a fixed smooth map where M can be considered as $M = \mathbb{C}^n$, $M = \mathbb{R}^n$ or a smooth compact manifold. Let a closed subset $A \in M$ with $f(A) = A$. If A is not stable in the sense of Lyapunov, it is called unstable.*

Definition 2.6 *Let $f : M \to M$ be a fixed smooth map where M can be considered as $M = \mathbb{C}^n$, $M = \mathbb{R}^n$ or a smooth compact manifold. We say it is asymptotically stable if it is Lyapunov stable and the realm of attraction is an open set.*

The concept of "realm of attraction" will be defined in the next chapter. Until then, it is worth noting that not all attractors are Liapunov stable, but, the most interesting examples of attractors usually are asymptotically stable.

2.4 Cantor Set

Some of the most interesting attractors found so far are Cantor sets, among them the Feigenbaum attractor, which will be studied in Chapter 5. Hence, it is worth to mention a section about the Cantor Set and its most important properties. To do so, firstly we give the formal definition of a Cantor set (together with some definitions needed for a well understanding of the concept), and secondly, an example is explained.

Definition 2.7 *A set is totally disconnected if all the connected components of the set are points.*

Definition 2.8 *A set is perfect if every point in it is an accumulation point or limit point of other points in the set.*

Definition 2.9 *A set A is a Cantor Set if it is closed, totally disconnected, and perfect subset of I.*

The Cantor Set, discovered by George Cantor, is a very important fractal set defined in the real interval $I = [0, 1]$. A very curious property about it is that it has zero measure but it is neither empty nor numerable. Let me build one as an example.

We start taking the interval $I = [0, 1]$, we subtract the open interval $(\frac{1}{5}, \frac{2}{5})$ from I, giving rise to two new intervals $I_1 = [0, \frac{1}{5}] \cup I_2 = [\frac{2}{5}, \frac{3}{5}]$. Repeating the same process we subtract the interval $(\frac{3}{5}, \frac{4}{5})$, giving rise to the last interval $[\frac{4}{5}, 1]$. Repeating the same steps subtracting the interior intervals of each resultant interval, we obtain numerous disjoint intervals inside I.

Since it can seem a bit complex to figure out, the cantor set of the example is given in figure 2.2.

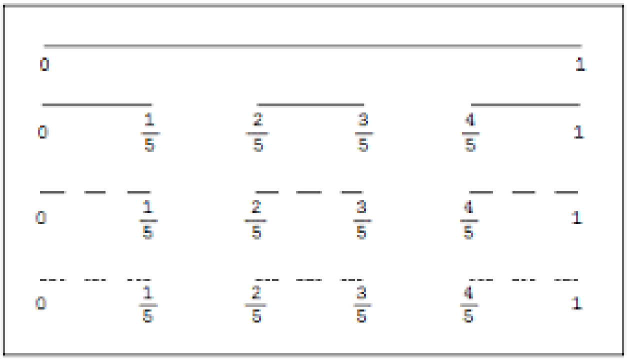

Figure 2.2: Cantor set.

For more information of the Cantor Set, you can take a look at [12].

Chapter 3

Definition of Attractor

3.1 First Definitions of Attractor

From now on, our main focus will be on the study of attractors. The concept of attractor has raised many different definitions since it appeared in the middle of the 20th century. However, it is important to remark that there is still no agreement on the most accurate definition of an attractor.

Coddingtong and Levinson ([2],1955), and later on, Auslander, Bhatia, and Seibert ([1],1964), were the first ones to use the concept of attractor as a compact invariant set M where all the orbits that stayed in a neighborhood of M approached M. The first ones only applied it to the case of a single invariant point while the second ones applied it to a set of invariant points. Even though this definition can seem clear to us, many experts at that time thought orbits that wander far away before converging back to M cannot be considered as attractors, and these cases were explicitly included in their definition. Hence, newer definitions appeared excluding these unstable cases. The most popular one was the"Axiom A attractor", defined by Smale ([10],1967).

This new and complex definition of attractors defined what we call nowadays Smale's attractors. Since the definition of Smale can seem complicated and unfamiliar to us, let us use the definition that Williams gave one year later to define the same objects in a simpler way.

Definition 3.1 *A subset A of $\Omega(f)$ is an attractor of f, provided it is indecomposable and has a neighborhood U such that f(U) $\in$ U and $\bigcap_{i>0} f^i(U) = A$.*

Remark. In this definition, $\Omega(f)$ denotes the non-wandering set of f which consists on all the non-wandering points $x \in M$ such that for every open set U containing x and $N > 0$, there is some $n > N$ such that $\mu(f^n(U) \cup U) > 0$ for some measure μ.

Although it is considered one of the most important definitions in the history of attractors, it did not stop the emergence of newer definitions of attractor.

Actually, the moment when Ruelle and Takens ([9],1971) suggested the existence of some kind of "strange attractors" causing the turbulent behaviour in fluids, the concept of attractor drew attention among the scientific community. The new definition they considered was:

Definition 3.2 *A closed subset A of the non-wandering set Ω is an attractor if it has a neighborhood U such that $\bigcap_{t>0} D_{X,t}(U) = A$, where $D_{X,t}$ denoted the flow on a smooth manifold generated by a vector field X.*

Nevertheless, it should be pointed out that Ruelle and Takens were not the first ones to consider this idea, since Lorenz made an experiment supporting the same idea few years before, even though he never gave an explicit definition of these kind of attractors.

In the following years, all of these definitions seemed very restrictive (most of the examples of attractors considered nowadays would not be an attractor following the previous definitions), therefore, it appeared newer definitions less restrictive. For example, Guckenheimer ([5],1976) defined an attractor as a fundamental system of neighborhoods, each of which is forward invariant under the flow generated by a vector field X. But this definition can seem too broad considering not every Liapunov stable set should be called an attractor.

Hence, between all the definitions existing in the literature, from now on we will consider a definition very near to the one given by Collet and Eckemann ([3],1980), which formally defines an attractor as the set of points to which most points evolve under iterations of a given map f.

3.2 Our definition of attractor

To start this section, let us give some indispensable definitions needed to understand the concept of attractor. We assume $M = \mathbb{R}^n$, $M = \mathbb{C}^n$ or a smooth compact manifold.

Definition 3.3 *We define the omega limit set $\omega(x)$ of a point $x \in M$ to the collection of all accumulation points of the sequence $\{x, f(x), f^2(x), ...\}$ of successive images of $x \in M$ given a smooth map f.*

If we choose some metric for M, then $\omega(x)$ can also be described as the smallest closed set S such that the distance from $f^n(x)$ to the nearest point of S tends to zero as $n \to \infty$. We should point out that $\omega(x)$ is always closed and nonvacuous, and also satisfies that $f(\omega(x)) = \omega(x)$. Moreover, $\omega(x)$ is always contained in the non-wandering set $\Omega(f)$ defined in the previous page.

Definition 3.4 *The realm of attraction $\rho(A)$ of a closed subset $A \subset M$ consists of all points $x \in M$ for which $\omega(x) \in A$.*

Once these concepts are clear, we give the formal definition of an attractor.

Definition 3.5 *A closed subset $A \subset M$ will be called an attractor if it satisfies two conditions:*

- *the realm of attraction $\rho(A)$ must have strictly positive measure (in other words, the probability of a point $x \in M$ falling into the realm of attraction must always be positive).*
- *there is no strictly smaller closed set $A' \subset A$ so that $\rho(A')$ coincides with $\rho(A)$ up to a set of measure zero.*

It should be pointed out that when we talk about "measure", we mean some measure μ on M equivalent to the Lebesgue measure. This can be constructed using a partition of unity, or using the volume form associated with a Riemannian metric.

Property 3.1 *Let A be an attractor, then $\rho(A)$ is necessarily a Borel set, and hence is measurable.*

As a consequence of this property is that any finite union of attractors is an attractor and the closure of an arbitrary union of attractors is also an attractor.

In conclusion, we can summarise the concept of attractor as a closed and non-vacuous subset $A \subset M$ contained in the non-wandering set $\Omega(f)$ satisfying $f(A) = A$ with f a continuous map from M to itself.

For those who would like to expand their knowledge on this subject, it should be noted that the realm of attractor may be called by different names in the literature. It is usually called the "basin of attraction" if it is an open set and the "stable manifold" if it is a lower dimensional smooth manifold.

We define two important attractors in the literature.

Definition 3.6 *A closed set $A \subset M$ is a minimal attractor if and only if its realm of attraction $\rho(A)$ has positive measure (considering a measure μ on M equivalent to the Lebesgue measure as before) and there is no strictly smaller closed set $A' \subset A$ for which $\rho(A')$ has positive measure.*

Some important properties about this kind of attractors are given below.

Property 3.2 *Let A be a minimal attractor, then $\omega(x)$ must be precisely equal to A for almost every x in the realm of attraction.*

Property 3.3 *The number of minimal attractors that can have a continuous function f is at most countably infinite.*

Given that minimal attractors exist, the question of whether there are also maximal attractors should come to mind. As expected, there exist some kind of maximal attractors called "the likely limit set".

Definition 3.7 *The likely limit set $\Lambda = \Lambda(f)$ is the smallest closed subset of M with the property that $\omega(x) \in \Lambda$ for every point $x \in M$ outside of a set of measure zero.*

Lemma 3.1 *The likely limit set Λ is well defined and is an attractor of f. In fact, Λ is the unique maximal attractor, which contains all others.*

The definition of attractor given in this section is the definition to the one we are referring from now on. Te previous ones are an attempt to show to the reader the many possible ways in which an attractor could be defined, as a consequence of the large number of attractors that have been discovered, some theoretically and some others experimentally. The easiest examples can be found in fixed points and periodic orbits, but also there are more difficult ones including the fascinating and intriguing strange attractors, about the ones we are including some information in the last chapter. Throughout this paper, we are dedicated to the study of some of these examples attending to our definition of attractor.

Chapter 4

Fixed points

In this chapter, we are going to study the different properties we can find about fixed and critical points in the discrete and continuous cases respectively. To do so, firstly we study the discrete case giving some basic definitions about fixed points together with some examples where we apply those definitions. Secondly, we study the continuous case also giving some basic definitions related to critical points and their most important properties, together with some examples for a better understanding of the problem.

4.1 Discrete Dynamical systems

We start this section giving the definitions of fixed and periodic points in discrete dynamical systems, together with some basic properties we are going to use in the study of the following examples.

Definition 4.1 *A fixed point x is a point that satisfy $f(x) = x$.*

Definition 4.2 *A periodic point x of period n is a point x that satisfies $f^n(x) = x$. In particular, for $n = 1$, we have a fixed point. The set of iterations of a periodic point form a periodic orbit.*

Maps with hyperbolic periodic points are the ones that occur typically in many dynamical systems, and therefore, it is important to analyze what is its behaviour.

Definition 4.3 *Let $f \in C^1$ and p be a periodic point of prime period n. We denote hyperbolic points to the points p which satisfy $|(f^n)'(p)| \neq 1$ and non-hyperbolic points to the points p which satisfy $|(f^n)'(p)| = 1$.*

Only hyperbolic points can be attractors according to the following definition:

Definition 4.4 *Let p be a hyperbolic periodic point of period n with $|(f^n)'(p)| < 1$. The point p is called an attracting periodic point.*

Definition 4.5 *Let p be a hyperbolic periodic point of period n with $|(f^n)'(p)| > 1$. The point p is called a repelling periodic point.*

But these are not the only attractors we can find. If we study the local behaviour of a dynamical system in the neighbourhood of a non-hyperbolic equilibrium point we can find another type of attractors called "weekly attracting points", which attracts points near them with a lower force.

However, it is important to keep in mind that non-hyperbolic points are unpredictable and their behaviour changes depending on the map we study. It can sometimes act like a weekly attracting point (and hence, it would be an attractor) but it can also act like a weekly repelling point or even it is possible that it is weekly repelling from one side and weekly attracting from the other.

Most maps have only hyperbolic periodic points, however, non-hyperbolic periodic points often occur in families of maps. When this happens, the periodic point structure often undergoes a bifurcation (defined in Chapter 5) and the subsequent unleashing of chaos (we will see two examples in the last chapters of the dissertation).

Example 4.1 *Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$.*

The fixed points of the function $f(x)$ are the same as the roots of the function $g(x) = f(x) - x$ which is $g(x) = x^2 - x$. Since the solutions of $x^2 - x = 0$ are $x = 0$ and $x = 1$, we have that both points are fixed points.

If we do the derivative $f'(x) = 2x$, we obtain $f'(0) = 0$ and $f'(1) = 2$ respectively. Therefore, we have both are hyperbolic fixed points and since the value of $f'(0)$ is lower than 1 we have $x = 0$ is an attracting fixed point.

Example 4.2 *Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin x$.*

The fixed points of the function $f(x)$ are the same as the roots of the function $g(x) = \sin x - x$. Since the root of $\sin x - x = 0$ is $x = 0$, we have it is a fixed point.

If we do the derivative $f'(x) = \cos x$, we obtain $f'(0) = 1$. Henceforth, it is a non-hyperbolic fixed point. To study the behaviour of the function near this point we do a graphical analysis in picture 4.1.

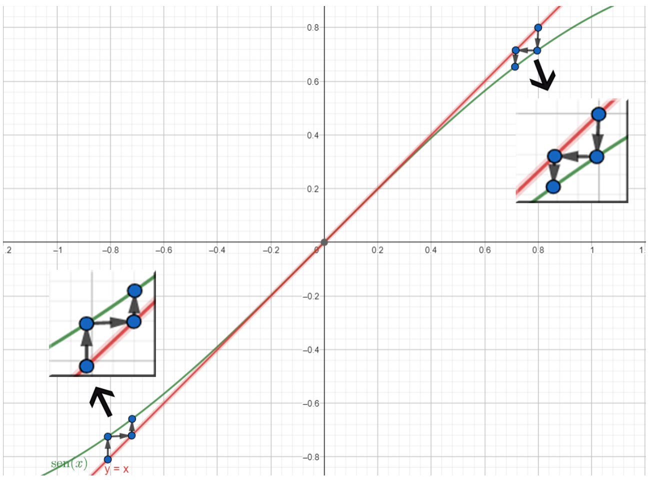

Figure 4.1: Graphic Analysis of $f(x) = \sin x$ in a neighbourhood of $x = 0$.

As we can see in the picture, taking random values from both sides triggers a series of iterations that approach slowly $x = 0$. Henceforth, we have a weekly attracting point.

Example 4.3 *Let $f : M \to M$, where M is the circle of real numbers modulo 2π defined as $f(\alpha) = \alpha + 1 - cos(\alpha)$ $(mod\ 2\pi)$.*

To find the fixed points of $f(\alpha)$ $(mod\ 2\pi)$ we look for the roots of the function $g(\alpha) = 1 - cos(\alpha)$ $(mod\ 2\pi)$. Since the only root is $\alpha = 0$, we conclude $\alpha = 0$ is the only fixed point of our function. All that remains for us to do is to demonstrate that it is indeed attractive.

Since $f(\alpha) \in C^\infty$, we take $f'(\alpha) = 1 + sen(\alpha)$ $(mod\ 2\pi)$. Evaluating $\alpha = 0$ we obtain $f'(0) = 1$. It is a non-hyperbolic fixed point, so it does not guarantee attraction. We need to do a graphical analysis to see the behaviour of the function at this point.

On the one hand, if we take as initial value $x = -4$ and we iterate the function several times, we see in figure 4.2 that the following iterations are getting closer to $\alpha = 0$, therefore, it is weakly attracting from the left. On the other hand, if we take as initial value $\alpha = 2$, we see in figure 4.2 that the following iterations are moving away from $\alpha = 0$, therefore, it is weakly repelling from the right.

About the stability of this map, we know it is not Liapunov stable since we cannot take a neighborhood sufficiently small of $\alpha = 0$ mapped into itself by f. Indeed, we can find a

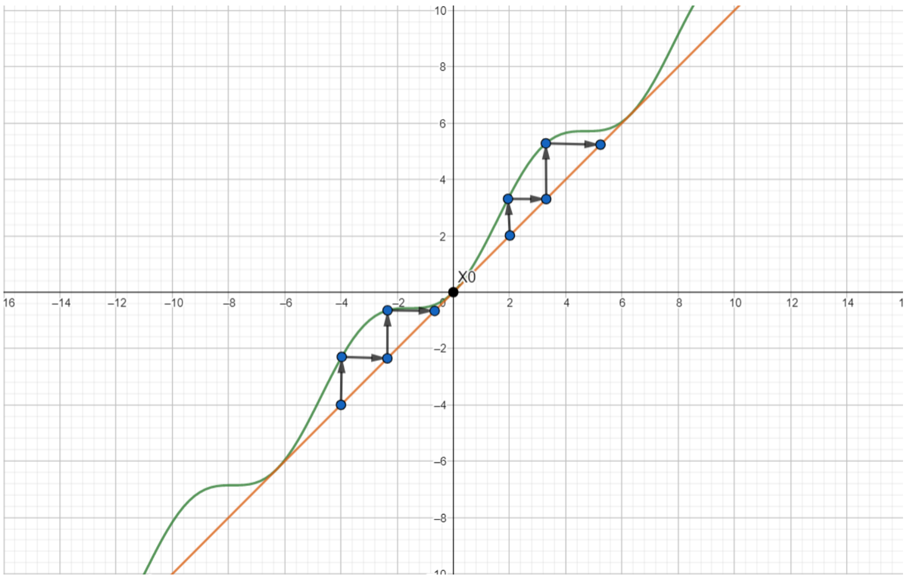

Figure 4.2: Graphic Analysis of $f(\alpha)$ in a neighbourhood of $x = 0$.

neighborhood U with such properties for points in the left side of $\alpha = 0$, but points in the right side will escape from U as they are iterated by f. In this case, we call $\alpha = 0$ an attractor which is one-sided stable.

Let me do a modification of the previous example.

Example 4.4 *Let $f : M \to M$, where M is the circle of real numbers modulo 2π defined as $f(\alpha) = \alpha + sin^2(\alpha)\ (mod\ 2\pi)$.*

To find the fixed points of $f(\alpha)\ (mod\ 2\pi)$ we look for the roots of the function $g(\alpha) = \sin^2(\alpha)\ (mod\ 2\pi)$ and we find two fixed points in $\alpha = 0$ and $\alpha = \pi$. Since $f(\alpha) \in C^\infty$, we take $f'(\alpha) = 1 + 2\, sen(\alpha)\, cos(\alpha)\ (mod\ 2\pi)$. Evaluating $\alpha = 0$ and $\alpha = \pi$ we obtain $f'(0) = 1$ and $f'(\pi) = 1$ respectively. Hence, both points are non-hyperbolic fixed points and it does not guarantee attraction. We need to do a graphical analysis to see the behaviour of the function at this point.

In figure 4.3 we observe that there are arbitrary points closed to either one whose successive images converge to the other. For example, points $0 < \alpha < \pi$ that are closer to $\alpha = 0$ than $\alpha = \pi$ converge to $\alpha = \pi$, and points $\pi < \alpha < 2\pi$ that are closer to $\alpha = \pi$ converge to $\alpha = 0\ (mod\ 2\pi)$. In this case, we say we have two attractors in $\alpha = 0$ and $\alpha = \pi$ which are not stable.

Example 4.5 *Let $f : \mathbb{R} \cup \infty \to \mathbb{R} \cup \infty$ defined as $f(x) = x + 1$.*

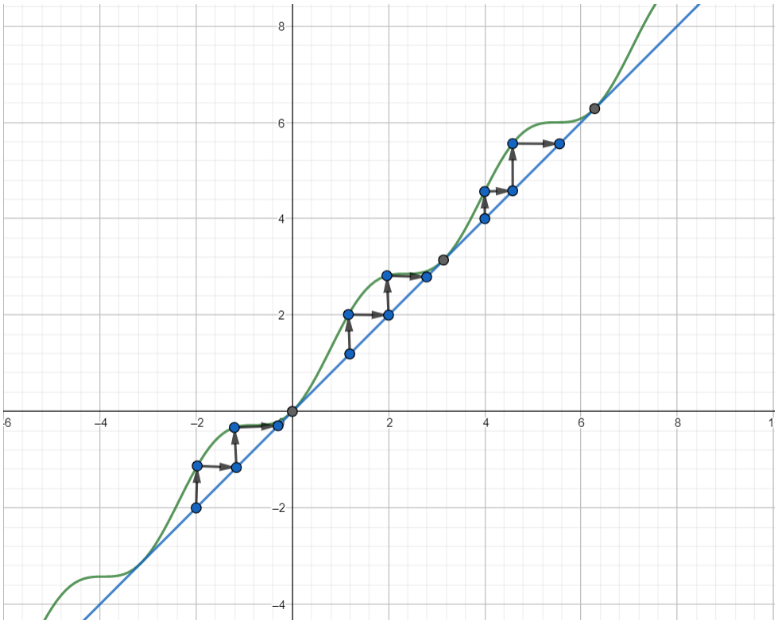

Figure 4.3: Graphic Analysis of $f(\alpha)$ in a neighbourhood of $\alpha = 0$ and $\alpha = \pi$.

It does not have any fixed points. Does it have any attractors? If we take as initial value a random value $x_0 = a \in \mathbb{R}$, $\lim_{n\to\infty} f^n(x) = \infty$. So, $x = \infty$ is the unique attractor of the function.

All of the attractors of these examples are in fact minimal attractors for obvious reasons.

Example 4.6 *Let* $f : \mathbb{R} \to \mathbb{R}$ *defined as* $f(x) = \frac{3\sqrt{3}(x-x^3)}{2}$

An important property of this function is that if $x \in I_1 = [-1, 0]$, then we have that $f(x) = \frac{3\sqrt{3}}{2}(x - x^3) \in I_1$, and more generally, $f^n(x) \in I_1$ for all $x \in I_1$. The same property is satisfied for $I_2 = [0, 1]$, therefore, we have that each of these intervals maps precisely into itself. If we take as initial value $x_0 \in I_i$ for $i = 1, 2$, we can see visually this property in figure 4.4.

Each of these intervals form two unstable minimal attractors, but the union of both intervals $I = [-1, 1]$ is asymptotically stable. And even more, the likely limit set of $f(x)$ consists of $\Lambda = [-1, 0] \cup [0, 1]$.

Now, if we take as initial value $x_0 \notin I$, we can see its image by f increase enormously as we take values outside the interval (this phenomenon is represented in figure 4.5). Hence, we have $\{\infty\}$ is an stable attractor.

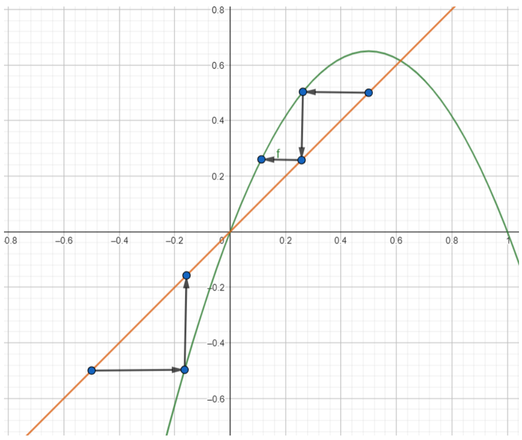

Figure 4.4: Graphic Analysis of $f(x)$ in $I_1 = [-1, 0]$ and $I_2 = [0, 1]$

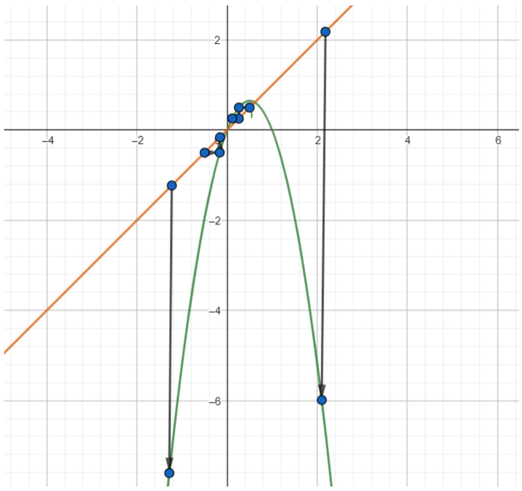

Figure 4.5: Graphic Analysis of $f(x)$ in $\{\infty\}$.

4.2 Continuous Dynamical Systems

In this section, we will focus on the study of equilibrium and periodic solutions. To do so, we start considering an important property about autonomous equations which is essential in the study of periodic solutions.

Lemma 4.1 *Suppose that we have a solution $\phi(t)$ of equation $\dot{x} = f(x)$ in the domain $D \subset \mathbb{R}^n$, then $\phi(t - t_0)$ with t_0 a constant is also a solution.*

Proof. Let $\gamma = t - t_0$ be a change of variable. If we use it in the equation $\dot{x} = f(x)$, the equation does not change since the variable t does not appear explicitly. Hence, if $\phi(t)$ is a solution of the equation, $\phi(\gamma)$ must be also a solution of the transformed equation. $\square$

Remark: Although the two solutions defined in the previous lemma correspond to different solutions of the equation, they correspond with the same orbital curves in the phase space.

Definition 4.6 *Let a point $a \in \mathbb{R}^n$ be a zero of the vector function $f(x) = (f_1(x), ..., f_n(x))$. We will call this point a critical point (sometimes it is also called equilibrium point).*

Example 4.7 *The harmonic oscillator: $\ddot{x} + x = 0$.*

Doing the change of variable $\{x = x_1, \dot{x} = x_2\}$, we obtain the autonomous equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1. \end{aligned}$$

If we define the following equation

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2},$$

then, the solutions of the equation provide us with the orbits of the phase-space

$$x_1^2 + x_2^2 = c \; (c \; \in \mathbb{R}).$$

This is a family of concentric circles in the phase-space, where the origin $(0, 0)$ is a critical point. See figure 4.6.

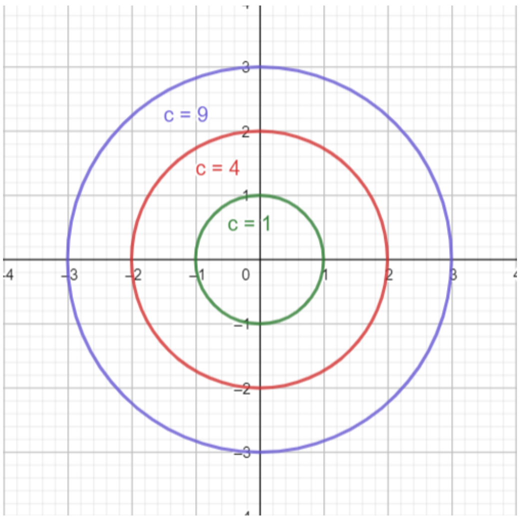

Figure 4.6: Graphic of the phase space of the harmonic oscillator $\ddot{x} + x = 0$.

Example 4.8 *Modified equation of the harmonic oscillator: $\ddot{x} - x = 0$.*

Doing the previous change of variable, we obtain the autonomous equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1. \end{aligned}$$

If we define the following equation

$$\frac{dx_2}{dx_1} = \frac{x_1}{x_2},$$

then, the solutions of the equation provide us with the orbits of the phase-space

$$x_1^2 - x_2^2 = c \ (c \ \in \mathbb{R}).$$

Even though the equation is very similar to the previous one, in this case we obtain a family of hyperboles in the phase-space, where the origin $(0, 0)$ is again a critical point. See figure 4.7.

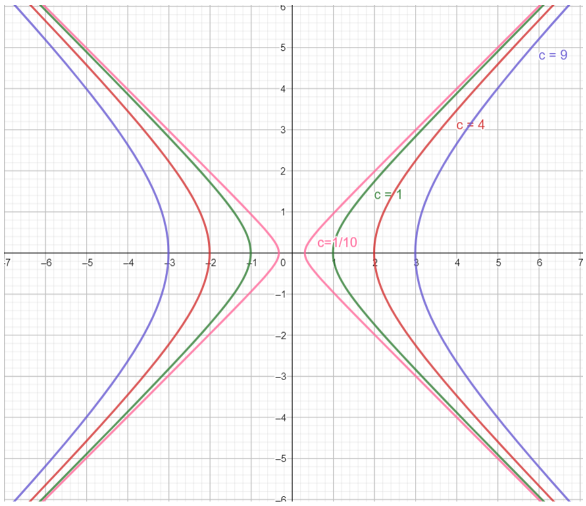

Figure 4.7: Graphic of the phase space of the modified harmonic oscillator $\ddot{x} - x = 0$.

Definition 4.7 *Let a solution $x(t)$ satisfy the autonomous equation for all time, i.e., $x(t) = a$ with $a \in \mathbb{R}^n$. It is usually called an equilibrium solution (or stationary solution).*

Remark. Critical points always corresponds with an equilibrium solution. Also, equilibrium solutions can never be reached in finite time, otherwise, the solutions would intersect (this is a result from the existence and uniqueness theorem given in [13], page 3).

Example 4.9 *Let* $\dot{x} = -x$, $x(0) = x_0$ *for* $t > 0$, $x_0 \neq 0$.

The solution of this initial value problem is $x(t) = x_0 e^{-t}$. We study the behaviour of the solution in the infinity, to do so, we do the limit of the function as follows,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x_0 e^{-t} = 0.$$

We can see $x = 0$ is an equilibrium solution for the system. Moreover, $x = 0$ is a critical point. See figure 4.8.

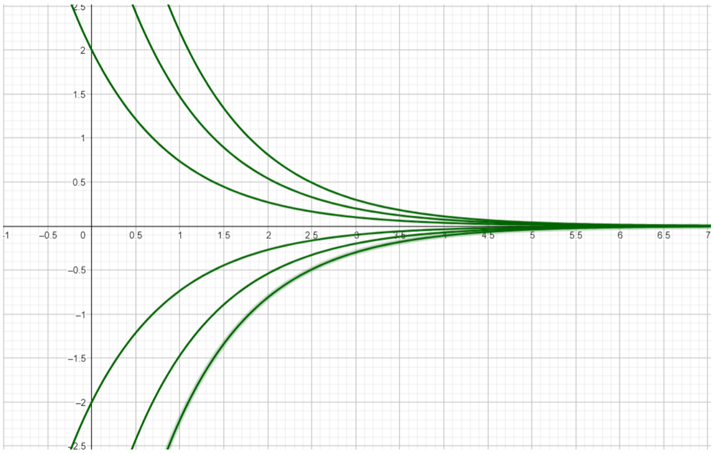

Figure 4.8: Graphic of the solutions for different initial values $x_0 \neq 0$ of the function $x(t) = x_0 e^{-t}$.

Example 4.10 *Let* $\dot{x} = -x^2$, $x(0) = x_0$, *for* $t \geq 0$, $x_0 \neq 0$.

The solution of this initial value problem is $x(t) = \left(\frac{1}{x_0} + t\right)^{-1}$. If we study the solution for $t \to \infty$, the solutions tend to the limit toward the equilibrium solution $x = 0$ as follows,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} = \left(\frac{1}{x_0} + t\right)^{-1} = 0.$$

Moreover, we can say the orbits in one-dimensional phase space tend towards the critical point $x = 0$. This phenomenon is called attraction. We can see it in figure 4.9.

Definition 4.8 *A critical point* $x = a$ *of the equation* $\dot{x} = f(x)$ *in* $\mathbb{R}^n$ *is called a positive attractor if there exists a neighbourhood* $U_a \subset \mathbb{R}^n$ *of* $x = a$ *such that* $x(t_0) \in U_a$ *implies* $lim_{t\to\infty} x(t) = a$. *If a critical point* $x = a$ *has this property for* $t \to -\infty$, *then* $x = a$ *is a negative attractor.*

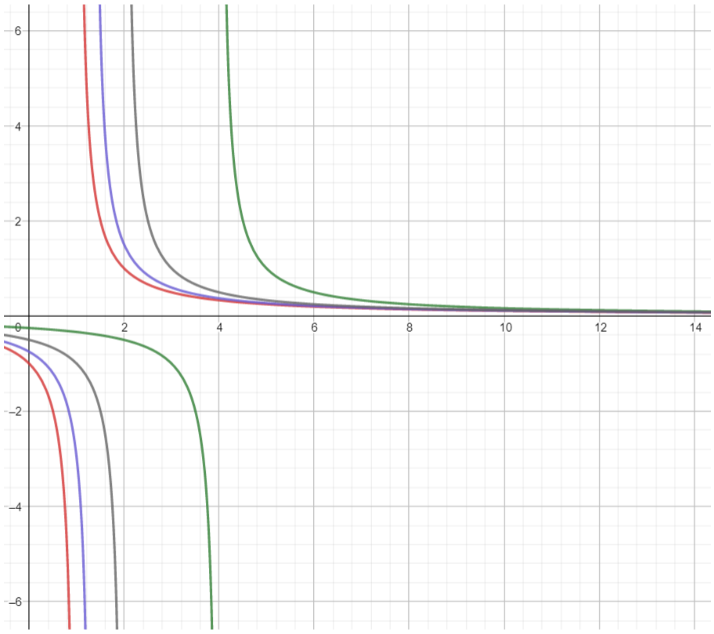

Figure 4.9: Graphic of the solutions for different initial values $x_0 \neq 0$ of the function $x(t) = \left(\frac{1}{x_0} + t\right)^{-1}$.

In analysing critical points and equilibrium solutions, we need to study the local behaviour of dynamical systems in the neighbourhood of each hyperbolic equilibrium point. To do so, the following theorem shows the topological structure of a non-lineal differential equation is the same than the linear system with $A = Df(x_0)$.

Theorem 4.1 ***Hartman-Grobman Theorem.*** *Let E be an open set of $\mathbb{R}^n$ that contains the origin, let $f \in C^1(E)$ and let ϕ_t the flow of the non-linear system. Suppose $f(0) = 0$ and the Jacobian matrix $A = Df(0)$ has eigenvalues with real part different from 0. Then, there exists an homemorphism $H : U \to V$ such that U and V contains the origin and for all $x_0 \in U$, there exists an open interval $I_0 \subset \mathbb{R}$ that contains the zero point and it is satisfied*

$$H \circ \phi_t(x_0) = e^{At}\, H(x_0) \quad \forall x_0 \in U, t \in I_0.$$

It means, H projects the trajectories from the non-linear system over the trajectories from the linear system in a neighborhood of the origin.

Hence, we can conclude from this theorem that every time we want to analyse critical points and equilibrium solutions we shall start always by linearising the equation in a neighbourhood of the critical point. That means, if we assume that $f(x)$ has a Taylor series expansion of terms of degree one plus other terms of degree higher than one, then, we only take the term of degree 1 in the neighborhood of the critical point $x = a$. So, in the case of an autonomous equation

$\dot{x} = f(x)$, we would study the linear equation $\dot{x} = \frac{\partial f}{\partial x}(a)(x-a)$, where $\frac{\partial f}{\partial x}(a)$ can be a matrix A_{nxn} with constant coefficients. Therefore, from now on, the linearised system which we will study will be denoted as $\dot{x} = Ax$, where we consider the case in which $|A| \neq 0$ (we assume the critical point is non-degenerate).

Also, it is important to point out that a consequence of this theorem is that all the critical points for which the jacobian matrix has eigenvalues of negative real part will be attractors. It does not mean that there cannot be attractors if this situations does not happen (it is not a necessary condition) but in the case it is happening, we will know we have an attractor at hand. There are also some theory that proof the existence of periodic orbits which are attractors but since this theory is out of the context of this chapter, we do not do a detailed study of this.

Example 4.11 *The Volterra-Lotka equations.*

Consider the system

$$\dot{x} = ax - bxy$$
$$\dot{y} = bxy - cy$$

with $x, y \geq 0$ and a,b,c positive constants. This system was formulated by Volterra and Lotka to describe the interaction of two species, where x denotes the population density of the prey, y the population density of the predator.

In this model the survival of the predators depends completely on the presence of prey, this means in a formal way that considering $x(0) = 0$ then, $\lim_{t\to\infty} y(t) = 0$ where $y(t) = y(0) \cdot e^{-ct}$ (it means, the predator would disappear).

The equilibrium solutions correspond with the critical points $(0,0)$ and $(\frac{c}{b}, \frac{a}{b})$. To study both cases, firstly, we put our system in the matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} ax - bxy \\ bxy - cy \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where $f(x, y) = (ax - bxy, bxy - cy)$. On the one side, doing the linearised system in a neighborhood of $(0,0)$ we obtain $\dot{x} = ax, \dot{y} = -cy$. Therefore, the solutions of the linearised form are $x(t) = x(0)e^{at}$ and $y(t) = y(0)e^{-ct}$. On the other side, doing the linearised system in a neighborhood of $(\frac{c}{b}, \frac{a}{b})$ we obtain $\dot{x} = -c(y - \frac{a}{b}), \dot{y} = a(x - \frac{c}{b})$. Therefore, the solutions of the linearised form are $x(t) = x_0 \cos(\sqrt{act}) + x_1 \sin(\sqrt{act})$ and $y(t) = y_0 \cos(\sqrt{act}) + y_1 \sin(\sqrt{act})$.

Definition 4.9 *Suppose that $x = \phi(t)$ is a solution of the equation $\dot{x} = f(x)$, $x \in D \subset \mathbb{R}^n$ and suppose that there exists a positive number T such that $\phi(t+T) = \phi(t)$, $\forall t \in \mathbb{R}^n$. Then, $\phi(t)$ is called a periodic solution of the equation with period T.*

Remark: Suppose T is the smallest period, then we call $\phi(t)$ T-periodic. Moreover, for a periodic solution we have that after a time T, $x = \phi(t)$ assumes the same value in $\mathbb{R}^n$. So, a periodic solution produces a closed orbit or cycle in the phase-space. This is formalized as follows,

Lemma 4.2 *A periodic solution of an autonomous equation $\dot{x} = f(x)$ corresponds with a closed orbit in phase-space and a closed orbit corresponds with a periodic solution.*

We have already seen a linear example where closed orbits exist (the harmonic oscillator), now we are going to see some nonlinear examples.

Example 4.12 *The mathematical pendulum.*

The pendulum can be described by the equation $\ddot{x} + \sin x = 0$. Doing the change of variables $\dot{x} = x_1$ and $\dot{x} = x_2$ we have,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 \end{aligned}$$

In this case the phase-space is two dimensional (see figure 4.10) and the orbits are described by the equation

$$\frac{dx_2}{dx_1} = -\frac{\sin x_1}{x_2},$$

whose integration yields

$$x_2^2 - 2\cos x_1 = c.$$

Once we have got the linearized transformation of the autonomous equation, we can characterize the different critical points we get depending on the eigenvalues and eigenvectors of the function. To do so, let me study first the cases we have for two-dimensional linear systems.

Since we are in dimension 2, we have two eigenvalues λ_1 and λ_2, which are real or complex conjugate. The general solution of our equation has to be of the form $x(t) = \begin{pmatrix} c_1\, e^{\lambda_1 t} \\ c_2\, e^{\lambda_2 t} \end{pmatrix}$ where c_1, c_2 are arbitrary constants.

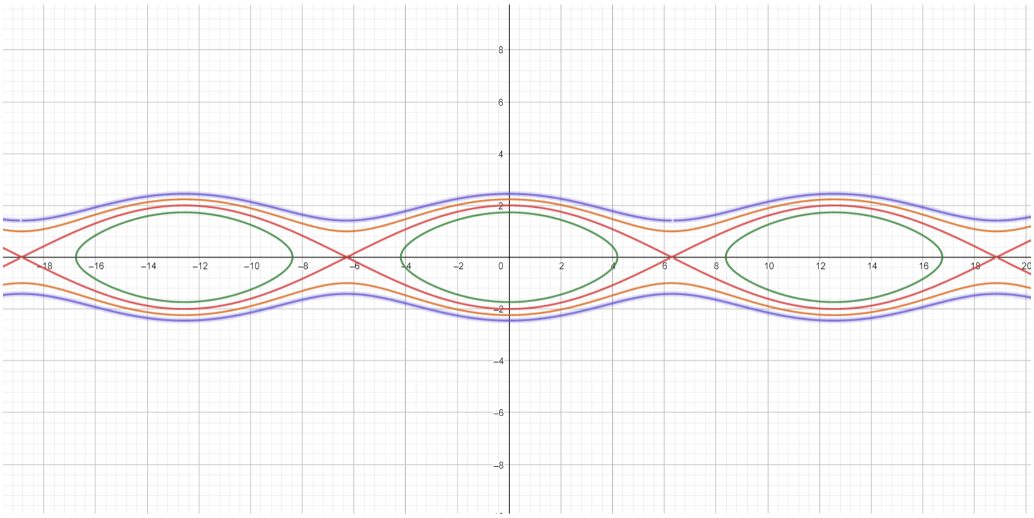

Figure 4.10: Graphic of phase space of the function $x_2^2 - 2\cos x_1 = c$ for different values of c.

In the case where the eigenvalues are real and have the same sign, we assume $\lambda_1 \neq \lambda_2$. Since the solutions are of the form $x_1(t) = c_1\, e^{\lambda_1 t}$ and $x_2(t) = c_2\, e^{\lambda_2 t}$, we can deduce the form of the orbits in the phase-space taking the modulo of both functions. Since the equality we obtain is $|x_1| = c\,|x_2|^{\frac{\lambda_1}{\lambda_2}}$, which are some kind of parabolas as we can see in figure 4.11.

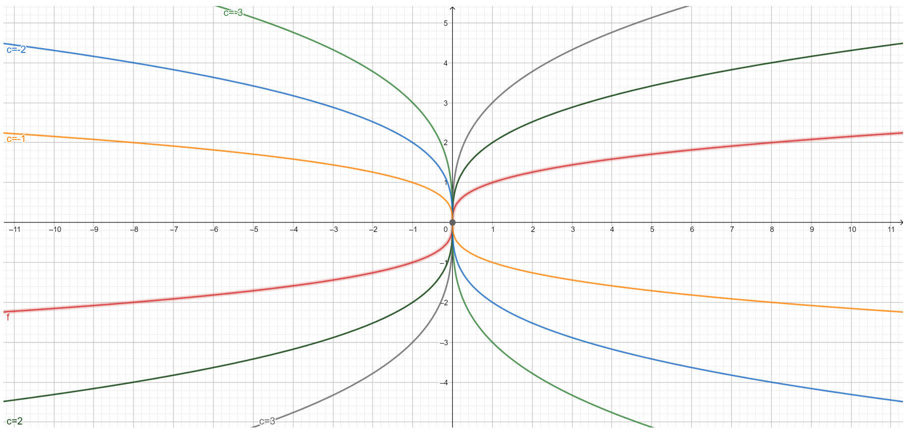

Figure 4.11: Graphic of the phase space of the function $|x_1| = c\,|x_2|^{\frac{\lambda_1}{\lambda_2}}$ for two real eigenvalues $\lambda_1 = -1$ and $\lambda = -3$.

The critical point we have in $(0, 0)$ is called a node. In the case $\lambda_1, \lambda_2 < 0$, we have a positive attractor. In the case $\lambda_1, \lambda_2 > 0$, we have a negative attractor.

In the case where the eigenvalues are real and have different sign, we have the solutions are also $x_1(t) = c_1\, e^{\lambda_1 t}$ and $x_2(t) = c_2\, e^{\lambda_2 t}$, therefore, we can deduce the form of the orbits in the phase-space taking the modulo of both functions. Since the equality we obtain is

$|x_1| = c\,|x_2|^{-\left|\frac{\lambda_1}{\lambda_2}\right|}$, we have some kind of hyperbolas as we can see in figure 4.12.

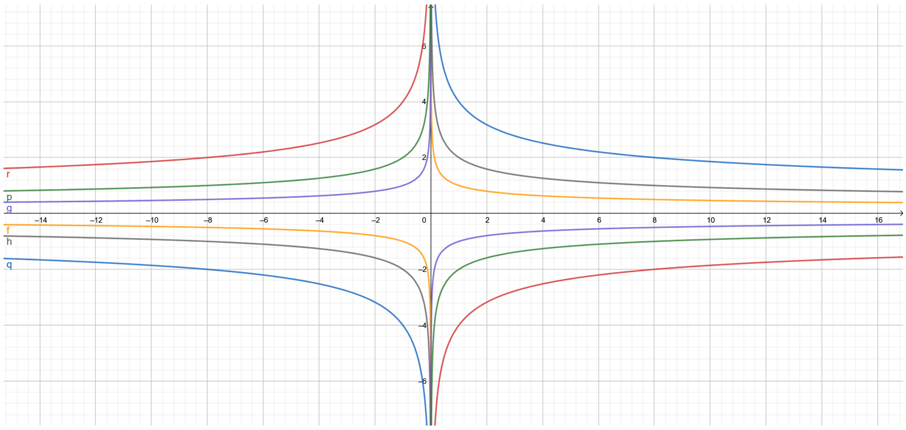

Figure 4.12: Graphic of the phase space of the function $|x_1| = c\,|x_2|^{-\left|\frac{\lambda_1}{\lambda_2}\right|}$ for two real eigenvalues $\lambda_1 = -1$ and $\lambda = 3$.

In this case the critical point $(0, 0)$ is a saddle point, and hence, it is not an attractor. Moreover, since we have one positive eigenvalue (assume $\lambda_1 > 0$) and another one negative (assume $\lambda_2 < 0$), there exist two solutions such that $(x_1(t), x_2(t)) \to (0, 0)$ for $t \to \infty$ which are called the stable manifolds of the saddle point and two solutions such that $(x_1(t), x_2(t)) \to (0, 0)$ for $t \to -\infty$ which are called the unstable manifolds.

In the case where the eigenvalues are complex conjugate $\lambda_{1,2} = \mu \pm wi$ with $\mu w \neq 0$, the complex solutions we obtain are $z_1(t) = e^{(\mu+wi)t}$ and $z_2(t) = e^{(\mu-wi)t}$, therefore, we can deduce from the definition of the complex exponential that $e^{\mu t}\cos(wt)$ and $e^{\mu t}\sin(wt)$ are also real solutions of our problem. This means, the orbits of the phase-space are spiralling in and out with respect to the critical point $(0, 0)$ (this is called a focus). In the case of spiralling in as in figure 4.13 (left side) the critical point is a positive attractor, this happens when $\mu < 0$. But in the case of spiralling out as in figure 4.13 (right side) the critical point is a negative attractor, this happens when $\mu > 0$.

The last special case we could have is when the eigenvalues are purely imaginary. It means, if $\lambda_{1,2} = \pm wi$. Then, we have the critical point $(0, 0)$ is a centre and the solutions can be written as a combination of $\cos(wt)$ and $\sin(wt)$. Moreover, if we draw the phase-space we can see the trajectories we have are circles. Hence, in this case the critical point $(0, 0)$ is not an attractor. See figure 4.14.

To end this section, we are going to introduce an important theorem in dynamical systems.

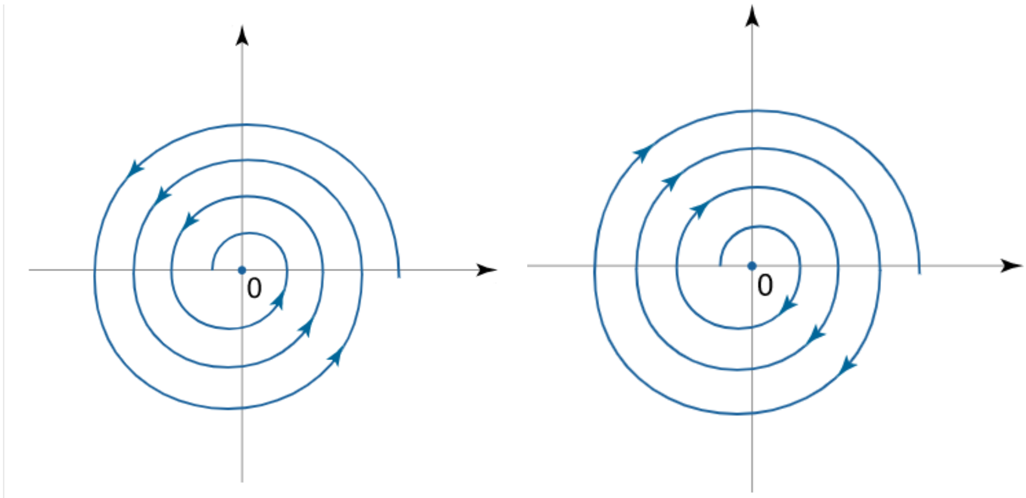

Figure 4.13: Graphic of phase space for two complex eigenvalues. On the left, we can see a stable focus. On the right, an unstable focus.

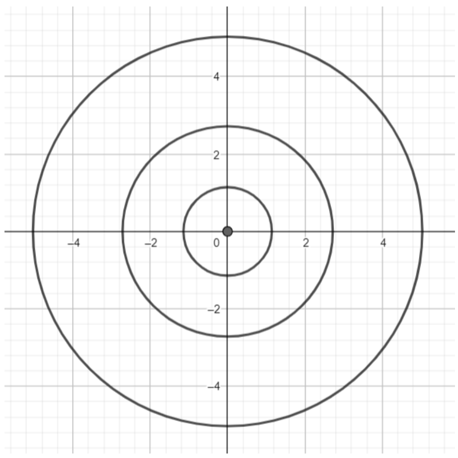

Figure 4.14: Graphic of phase space for two eigenvalues purely imaginary.

Theorem 4.2 *Consider the equation $\dot{x} = Ax + g(x)$, $x \in \mathbb{R}^n$. The constant $n \times n$-matrix A has n eigenvalues with nonzero real part, $g(x)$ is smooth (i.e. $g \in C^1$) and*

$$\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

Then, in a neighbourhood of the critical point $x = 0$, there exist stable and unstable manifolds W_s and W_U with the same dimensions n_s and n_u as the stable and unstable manifolds E_s and E_U of the linearised system $\dot{x} = Ax$. Moreover, in $x = 0$, E_s and E_U are tangent to W_s and W_U.

Proof. Since the proof is far from the purpose of our work, some references where you can find it are given: Hartman(1964), Chapter 9 and Knobloch and Kappel (1974), Chapter 5.

Example 4.13 *Dynamical system in* $\mathbb{R}^2$.

$$\left.\begin{array}{rcl}\dot{x} & = & -x \\ \dot{y} & = & 1-x^2-y^2\end{array}\right\}$$

To find the critical points of the system we define the vector function $f(x,y) = (-x, 1-x^2-y^2)$. The critical points are the points (x,y) that satisfy $f(x,y) = 0$, which means,

$$(-x, 1-x^2-y^2) = (0,0) \rightarrow \begin{array}{rcl} x & = & 0 \\ 1-y^2 & = & 0 \rightarrow y = \pm 1. \end{array}$$

Hence, the critical points we have are $(0,1)$ and $(0,-1)$. To characterize them we need to calculate the Jacobi matrix of the function $f(x,y)$,

$$J_{f(x,y)} = \begin{pmatrix} -1 & 0 \\ -2x & -2y \end{pmatrix}$$

If we substitute $(x,y) = (0,1)$ in this matrix and we calculate the eigenvalues, we obtain $\lambda_1 = -1$ and $\lambda_2 = -2$, so, doing the analysis previously studied we have a node which is a positive attractor. Now, if we substitute $(x,y) = (0,1)$ in the Jacobian matrix, we obtain as eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, so, we have a saddle point (which is not an attractor). Also, the conditions of the theorem 4.2 has been satisfied, therefore, there exist stable and unstable manifolds W_S and W_U with the properties defined in the theorem.

Remark: the stable manifolds of the saddle point separate the phase-plane into two domains, where the behaviour of the orbits is qualitatively different. Such a manifold we call a separatrix. In numerical calculations of stable and unstable manifolds it is convenient to start in a neighborhood of the saddle in points E_S and E_U, which have been obtained for the linear analysis (in the special case of example 3.1 we have $E_U = W_U$). In the case of a stable manifold we are integrating of course for $t \leq t_0$.

Chapter 5

The Feigenbaum Attractor

In this chapter we are going to do a detailed review about the Feigenbaum Attractor, an attractor which reveals one of the most important features of the quadratic functions, the transition from stable periodic behaviour to chaotic behaviour. The study will be initially focused on the real space, and later on we will extend it to the complex space.

In order to start the study of the quadratic function, it is essential to understand firstly the concept of bifurcation. A bifurcation means a split in two different parts, a change in the dynamical system. These changes often involve the periodic point structure, but may also involve other changes as well.

To understand how and when the periodic point structure of the family changes, we need to know which bifurcation occurs in the map considered. In the case of the quadratic family, it occurs a period-doubling bifurcation. Dynamically, the period-doubling bifurcation involves a change from an attracting to a repelling fixed point, together with the birth of a new period 2^n, $n \in \mathbb{N}$ orbit. This phenomenon is discussed below in the real space.

5.1 The real quadratic family

It is considered the function $Q_c : \mathbb{R} \to \mathbb{R}$ defined as $Q_c(x) = x^2 + c$. If we take the value $c = \frac{1}{4}$, we can observe on figure 5.1 a non-hyperbolic attracting fixed point in $x = \frac{1}{2}$. But if we decrease this value until $c = -\frac{3}{4}$ and $c = -\frac{5}{4}$, we can see in figure 5.2 the fixed point bifurcates into an attracting periodic orbit of period 2 and an attracting periodic orbit of period 4 respectively. In fact, if we keep decreasing the value of c, we can observe a sequence of period-doubling bifurcations $\{c_1, ..., c_n\}$ emerge very quickly up to a point where they converge (see the bifurcation diagram on figure 5.3). This point is the value $c_0 = -1,4011551890...$ the so-called Feigenbaum point. In particular, this first ten elements of the sequence $\{c_1, ..., c_n\}$ of period doubling bifurcations occurs for the values of c shown in Table 5.1.

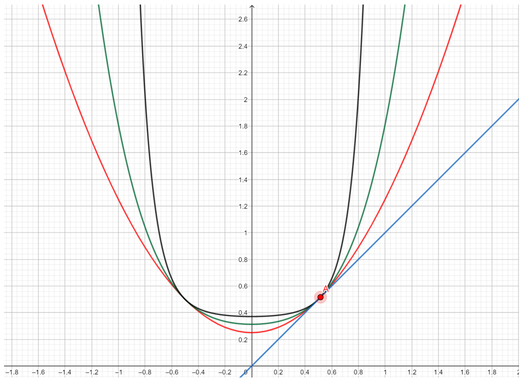

Figure 5.1: Graphic of the functions $y = Q_{0.25}(x)$ (red line), $y = Q^2_{0.25}(x)$ (green line) and $y = Q^4_{0.25}(x)$ (black line). The red point is the fixed point of $y = Q_{0.25}(x)$.

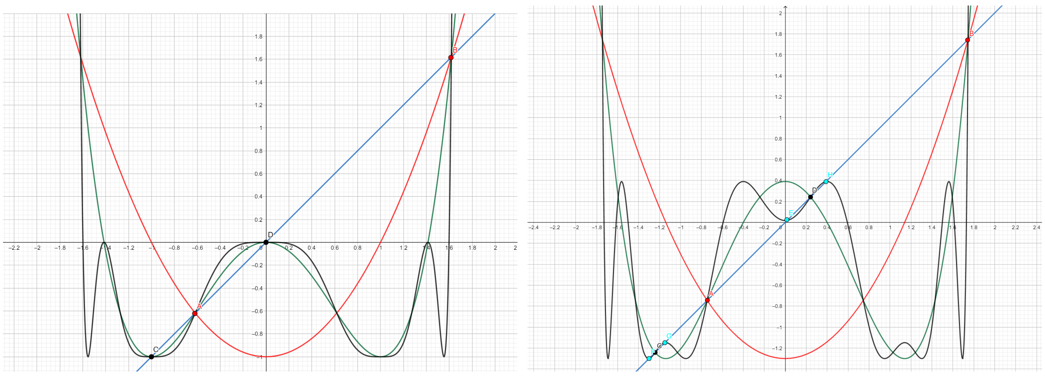

Figure 5.2: On the left, we have the graphic of the functions $y = Q_{-1}(x)$ (red line), $y = Q^2_{-1}(x)$ (green line) and $y = Q^4_{-1}(x)$ (black line). On the right, we have the graphic of the functions $y = Q_{-1.30}(x)$ (red line), $y = Q^2_{-1.30}(x)$ (green line) and $y = Q^4_{-1.30}(x)$ (black line). The black points are the points of period 2 we have and the light blue points are the points of period 4. As we can see, there appear new periodic points as we decrease the values of c.

Also it is known that the ratio of convergence is exponential as we can see in the following property.

Property 5.1 *The sequence of period doubling bifurcations* $\{c_1, ..., c_n\}$ *satisfies the equality* $c_n - c_0 \approx C\rho^{-n}$ *for* $n \in \mathbb{N}$ *with constant* $\rho = 4,6689...$

Initially, the value of ρ could be considered a completely casual value, but after doing some observations, it was found that the quadratic function was not the only function where this ratio of convergence appeared. For example, if we take the family of functions $f : \mathbb{R} \to \mathbb{R}$

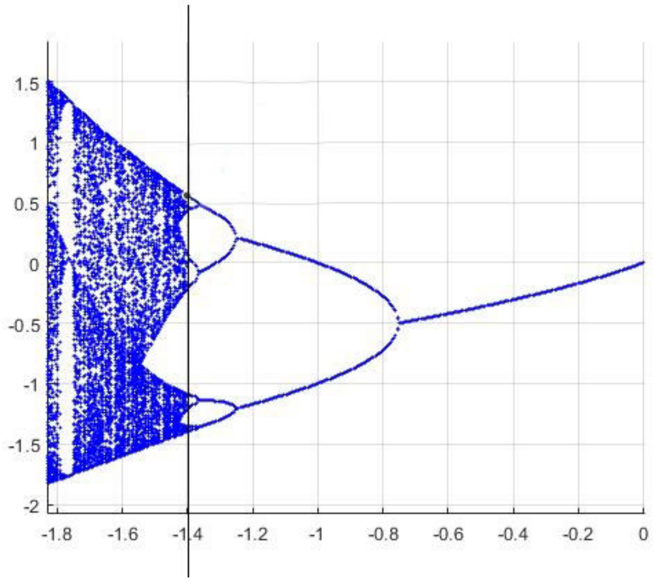

Figure 5.3: Bifurcation diagram of the quadratic function $Q_c(x) = x^2 + c$. A bifurcation diagram shows how bifurcations occur as the value of c decrease in a simple graphic. We plot the location of fixed and periodic points at the Y-axis versus the parameter of c at the X-axis of the graphic. A line is drawn at $c = -1.40$ to show in a more visual way the change in the behaviour of the dynamical system at the Feigenabum point (from stability to chaos).

defined as $f_b(x) = b \sin x$ for $b \in [0, \pi]$, we observe a similar sequence of period-doubling bifurcations $\{b_1, ..., b_n\}$ exponentially converging to a limit point b_0 with rate $\|b_0 - b_n\| \approx C\rho^{-n}$ with $\rho = 4,669$ (for more information, see [6]). Therefore, one would think the rate of convergence appears to be universal, independent of the particular family of maps under consideration.

In order to formalize what was happening in a mathematical definition, Feigenbaum and Coullet-Tresser formulated a conjecture (which later on became a theorem) that would completely explain the above universality.

Theorem 5.1 *Let us consider an infinite-dimensional space U of unimodal maps, and consider the doubling renormalization operator R in this space (for more information about this operator see [6], page 1046). It satisfies:*

- *R has a unique fixed point f_0 with an appropriate scaling factor μ.*
- *R is hyperbolic at this fixed point, that is, there exist two transverse R-invariant manifolds W^S and W^U such that the orbits $\{R^n f\}$ with $f \in W^S$ exponentially converge to f_0, while the orbits $\{R^n f\}$ with $f \in W^U$ are exponentially repelled from f_0.*
- *Dim $W^U = 1$.*

n	Period $= 2^n$	Bifurcation parameter c_n	ratio ρ
1	2	-0.75	–
2	4	-1.25	–
3	8	-1.3680989	4.2337
4	16	-1.3940462	4.5515
5	32	-1.3996312	4.6458
6	64	-1.4008287	4.6639
7	128	-1.4010853	4.6682
8	256	-1.4011402	4.6689
9	512	-1.401151982029	4.66899
10	1024	-1.401154502237	4.668999

Table 5.1: Bifurcation parameters c_n for $n = 1, ..., 10$.

- *W^U transversely intersects the doubling bifurcation locus B_1, where an attracting fixed point bifurcates into an attracting periodic orbit of period 2.*

A sketch of the proof can be seen on [6], page 1050. Since the doubling bifurcations loci B_n of higher periods (from 2^n to 2^{n+1}) are obtained by taking preimages of B_1 under R^n, we can deduce that any one-parameter family of unimodal maps that is transverse to W^S intersects the B_n at the points b_n. These points have the property of converging exponentially to a limit point $b_0 \in W^S$, where the rate of convergence ρ is the unstable eigenvalue of the jacobian matrix $A = DR(f_0)$, which is independent to the family we are considering.

To do the study of the quadratic function in the real space we take the Feigenbaum point c_0 previously considered, which is the smallest real value of c for which the function $Q_c(x)$ has infinitely many distinct periodic orbits. Following the definition of attractor from section 3.2, to be able to describe the Feigenbaum attractor, firstly we need to restrict our function into a function $Q : M \to M$ where M must be a smooth compact manifold. To do so, we look for a compact domain containing the origin, for example, the interval $I = [-1, 1]$. Therefore, from now on, we are considering the function $Q : I \to I$ defined as $Q(x) = x^2 + c_0$ which satisfies $Q(I) \subset I$.

To be able to describe the dynamic structure of the function, let me take an initial value $x_0 \in I$ which allows me to construct the successive iterations x_n given by $x_n = Q^n(x_0)$. It is satisfied that the succession of points $\{x_0, x_1, ..., x_n\}$ converge towards a Cantor set A, to demonstrate this affirmation is true, let me take the value $x_0 = 0$ (remember we took an interval I containing the origin) and we build the orbit $O^+(0)$ as described in Definition 2.2.

$$O^+(0) = \{0, -1.40115518, 0.56208065, -1.08522052, -0.22345160, -1.35122456, \\ 0.42465263, -1.22052532, 0.08852682, -1.39331818, 0.540179869, ...\}$$

As we can see above, $O^+(0)$ is an almost periodic sequence where all the numbers are different (if we look closely, $Q^5(0)$ is near to $Q^1(0)$, and $Q^9(0)$ is even nearer to $Q^1(0)$. Same for $Q^2(0), Q^6(0)$ and $Q^{10}(0)$ and so on...). This is just an example of what is happening in the long term.

Taking two random elements from the orbit $Q^n(0)$ and $Q^m(0)$ such that $m \neq n$. We have the difference $Q^m(0) - Q^n(0)$ is very small whenever $m - n$ is divisible by a power of 2. The higher the power of 2 is, the smaller will be the difference between them. Thus, we have the closure of this orbit (let me call it A) is a Cantor set, homeomorphic to the ring $lim(\frac{\mathbb{Z}}{2^k\mathbb{Z}})$ of 2-adic integers in such a way that each $Q^n(0)$ corresponds to a 2-adic integer n (i.e. each value $Q^n(0)$ can be written as $Q^n(0) = \sum_{i=0}^{n} a_i\, 2^i$).

Henceforth, we have already proven there exists a Cantor set A. And since the iterations $x_n = Q^n(x_0)$ converge towards A, we can conclude A is an attractor, and not any attractor, it corresponds to the likely limit set $\Lambda(f)$. A picture of the attractor is given by figure 5.4.

Now, if we consider the restriction to $Q : A \to A$, we have an homomorphism which adds one to each 2-adic integer (i.e. each value $x \in A$ can be written as $x = \sum_{i=0}^{n} a_i\, 2^i + 1$), from which we can conclude that all the points of A are different from each other and therefore we do not have periodic points inside. However, we can prove there are periodic points arbitrarily close to every point of A whose period is $n = 2k$, with $k \in \mathbb{R}$. All of these periodic points are unstable.

There are also a countable infinity of points $x \in I$ whose successive iterations do not converge towards A. Furthermore, these exceptional points are everywhere dense in I (although the tendency to converge to A remains strong) which leads to an important characteristic about the stability of the attractor. According to Guckenheimer [5], if Q has no periodic attractor, any arbitrarily small open interval $U \subset I$ has some forward image $Q^n(U)$ which contains the origin. If we consider a point x in U which maps to a periodic point in the neighborhood of the origin, we see that $\omega(x)$ is a periodic orbit disjoint from A. Then, we have that A is certainly not asymptotically stable.

However, just because it is not asymptotically stable does not mean that it cannot be Liapunov stable. In fact, if we take the intersection of a nested sequence of asymptotically stable sets we have that indeed A is Liapunov stable. A sketch of the proof is given below.

If U is an arbitrary small neighborhood of A, let me choose a smaller neighborhood V so that $f(V) \subset V$. If we start at any point x_0 in I and we obtain the successive iterations $x_n = Q^n(x_0)$ it can happen two things depending on the initial point x_0 we take.

1. This orbit eventually hits the open set V and remains trapped in $V \subset U$ forever after.
2. It manages to precisely hit one of the finitely many periodic points which lie outside of V, and remains trapped in a finite periodic orbit outside of V thereafter.

And this would conclude that A is indeed Liapunov stable.

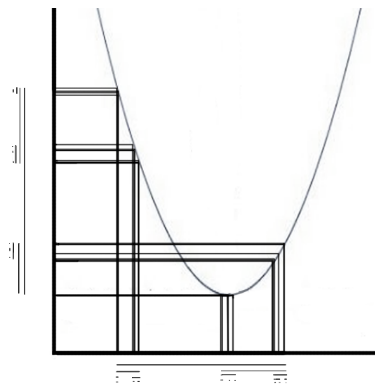

Figure 5.4: The Feigenbaum Attractor on the defined Cantor Set.

A simulation of the quadratic function $Q(x) = x^2 + c_0$ has been made using Matlab to visualize better the conclusions obtained. The code considered and a simulation taking as initial value $x = 0$ are given in figure 5.5.

We can see in the previous picture that the iterations of the function remain inside the interval $[-1.5, 0.5]$, therefore, we can deduce it remains trapped in a neighborhood of $x = 0$ forever after.

If we do other simulations considering other values x_0 as the initial condition (values near to $x = 0$, in particular, we consider initial values x_0 contained in the interval $I = [-1, 1]$) , we can see the iterations of the function remain inside of a neighborhood of x_0 too. To visualize this situation, we simulate the quadratic function $Q(x) = x^2 + c_0$ taking as initial values $x_0 = 0.5$

```
maxIter=100; %number of iterations
i = 1;
x(i) = 0; % initial value
y(i) = (x(i)^2)-1.401155189;
while i<=maxIter
    f= (x(i)^2)-1.401155189; %function
    i=i+1;
    x(i) = f;
    y(i) = f;
end

plot(y,'k')
title('Fiegenbaum Attractor')
xlabel('Abscissa axis')
ylabel('Ordinate axis')
```

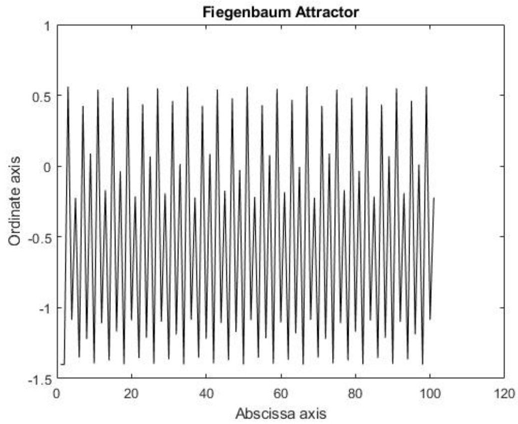

Figure 5.5: Simulation of the Feigenbaum attractor taking $x = 0$ as initial value.

and $x_0 = -0.5$, an the graphics we obtain for both of them are given in figure 5.6.

On the other side, if we take values of $x_0 \notin I$, the simulation goes to infinity very quickly. To visualize it, it is simulated the quadratic function considering $x_0 = 5$ and $x_0 = 50$, whose respective graphics are given in figure 5.7.

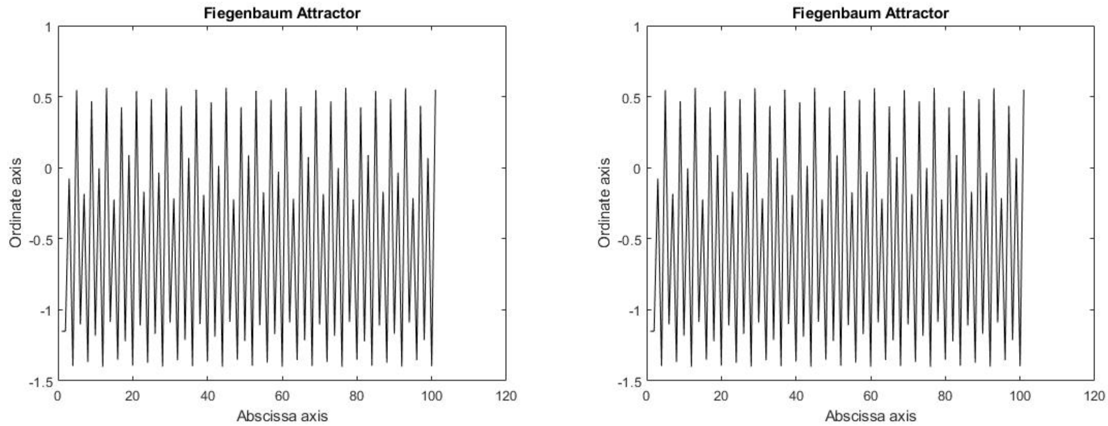

Figure 5.6: On the left, we can see a simulation of the Feigenbaum attractor taking as initial value $x_0 = +0.5$. On the right, the same simulation taking as initial value $x_0 = -0.5$

.

With this simulation we have been able to verify experimentally that the closure of the interval I chosen previously in theory, i.e. the Likely Limit Set Λ, acts as an attractor, while if we move away from the interval the function tends to $\{\infty\}$.

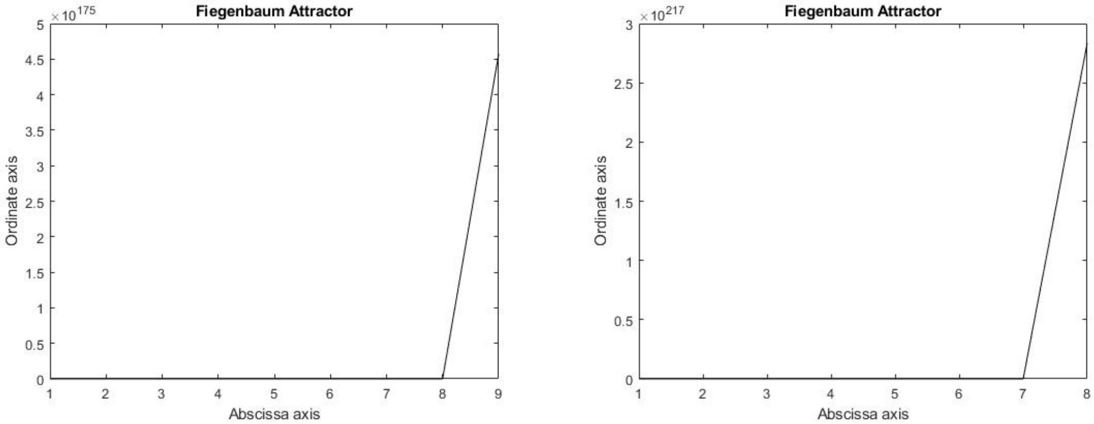

Figure 5.7: On the left, we can see a simulation of the Feigenbaum attractor taking as initial value $x_0 = 5$. On the right, the same simulation taking as initial value $x_0 = -50$.

5.2 The complex quadratic family

If we expand the study to the complex quadratic maps we can find more interesting properties which are worth to mention. For example, if we consider the function $Q_c : \mathbb{C} \to \mathbb{C}$ defined as $Q_c(z) = z^2 + c$ with arbitrary value $c \in \mathbb{C}$, it is always satisfied that the orbits placed near $\{\infty\}$ will escape to ∞ (something we have also seen for the real case). Hence, we can say we have a stable attractor in $\{\infty\}$, whose basin of attraction is defined as the set of all escaping orbits denoted as:

$$D_f(\infty) = \{z : f^n z \to \infty \; s.t. \; n \to \infty\}.$$

However, the really fascinating properties of quadratic functions are to be found in the complementary of this set, the so-called filled Julia Set, denoted as $K(f) = \mathbb{C} \setminus D_f(\infty)$, and its boundary the so-called Julia Set $J(f) = \partial K(f)$, which we will consider from now on. An example of the Julia set is given in figure 5.8, from which we can deduce the much richer topological and geometric structure that have complex quadratic functions compared to the real case.

To build The Julia Set, we use the following algorithm:

1. Choose an equal size grid in the square $[-2, 2] \times [-2, 2]$.

2. Given $N > 0$, it is defined the grid points using the following expression

$$z_{l,m} = -2 + 4\frac{l}{n} + \frac{-2 + 4m}{N}i, \quad forl, m = 0, ..., N.$$

3. For every complex number $z_{l,m}$ in the grid, it is checked if it escapes or not under Q_c. Let

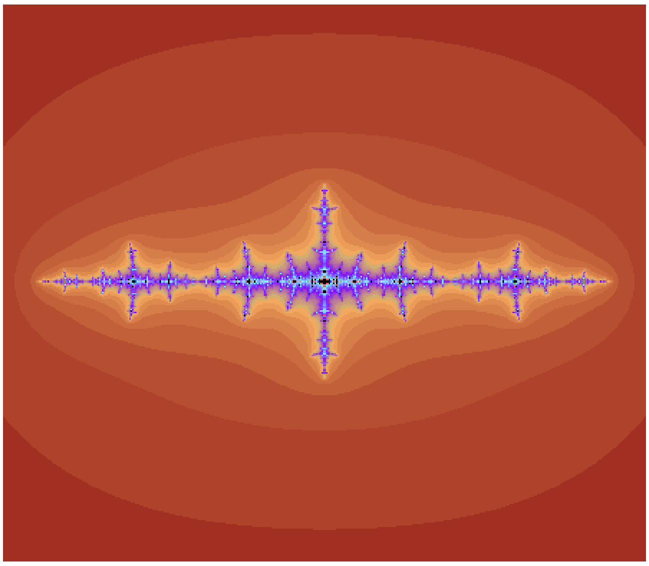

Figure 5.8: The Julia Set of the Fibonacci Attractor $f(z) = z^2 - 1.401$ from $z = -2 - 2i$ to $z = 2 + 2i$ [4].

$M > 0$, it is computed the first M iterates of $z_{l,m}$ under Q_c.

4. If, at any iteration $k < M$, $\|Q_c^k(z_{l,m})\| > 10$, we stop the iteration since it means the point $z_{l,m}$ escapes to ∞. If $\|Q_c^k(z_{l,m})\| < 10$ for all $k < M$, then it is assumed that $z_{l,m}$ belong to the filled Julia Set of Q_c.

The structure of the Julia Set can be either Cantor or connected depending on whether the critical point 0 escapes to infinity or not. In the cases where the Julia Set is connected (the ones we are considering in this paper), there exists a Mandelbrot set M in the c-plane which looks like in figure 5.9. For this figure, it is considered the parameter values $c \in I = [-2, \frac{1}{2}]$ where it is satisfied $I = M \cap \mathbb{R}$. It is clear that the Julia Set is connected for these values of c, in other case, we would not have been able to draw the Mandelbrot Set. Another example of the Mandelbrot set would be considering the parameter values $c \in I = [-1.25 - 0.1i, -1.5 + 0.1i]$, which is closer interval to the Feigenabum point. We can see how is the Mandelbrot Set on the left of figure 5.10. And even we can take a look closer to the Feigenbaum point considering the parameter values $c \in I = [-1.35 - 0.05i, -1.45 + 0.05i]$. In this case, the Mandelbrot Set is given on the right side of 5.10. A formalized definition of the Mandelbrot set would be the following:

Definition 5.1 *The Mandelbrot set is the subset of the c-plane given by*

$$M = \{c \mid Q_c^n(0) \nrightarrow \infty\}.$$

Equivalently, $M = \{c \mid K(c)$ *is connected*$\}$.

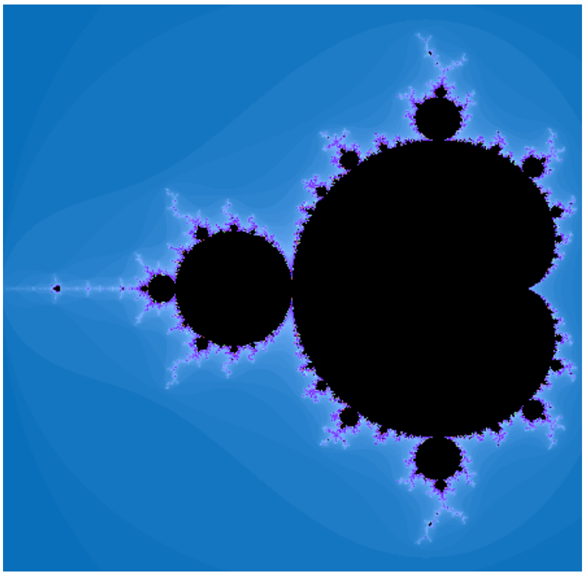

Figure 5.9: The Mandelbrot Set of the set of maps $f(z) = z^2 + c$ with $c \in I = [-2, \frac{1}{2}]$. [4].

To build The Mandelbrot Set, we use the following algorithm:

1. Let $c \in \mathbb{C}$, we simply compute the first N points on the orbit of c under Q_c.

2. If, at any iteration $k < N$, $\|Q_c^k(c)\| > 2$, we stop the iteration because $c \notin M$. If $\|Q_c^i(c)\| \leq 2$ for all $i \leq N$, we assume that $c \in M$.

From this algorithm we observe that M is contained inside the disk of radius 2 in the c-plane since $Q_c^n(c) \to \infty$ for $\|c\| > 2$, and so, these values of c does not belong to M. Also, if $\|Q_c^k(c)\| > 2$ for some $k \geq 0$, it similarly follows that $Q_c^n(c) \to \infty$.

Moreover, if we look closer of figure 5.9, we can notice the large cardioid-shaped region in the center. This main cardioid is the region of parameters c for which the quadratic function $Q_c(x)$ has an attracting fixed point. It consists of all parameters of the form

$$c = \frac{\mu}{2}(1 - \frac{\mu}{2})$$

for some μ in the open unit disk. To the left of the main cardioid, attached to it at the point $c = -\frac{3}{4}$, a circular-shaped bulb is visible. This bulb consists of those parameters c for which f_c has an attracting periodic orbit of period 2. The set of parameters c is a circle of center $c = -1$ and radius $r = \frac{1}{4}$. And we could keep doing this forever, since there are infinitely many other bulbs tangent to the main cardioid. A correspondence between the Mandelbrot set and the bifurcation diagram of the quadratic map is in figure 5.11.

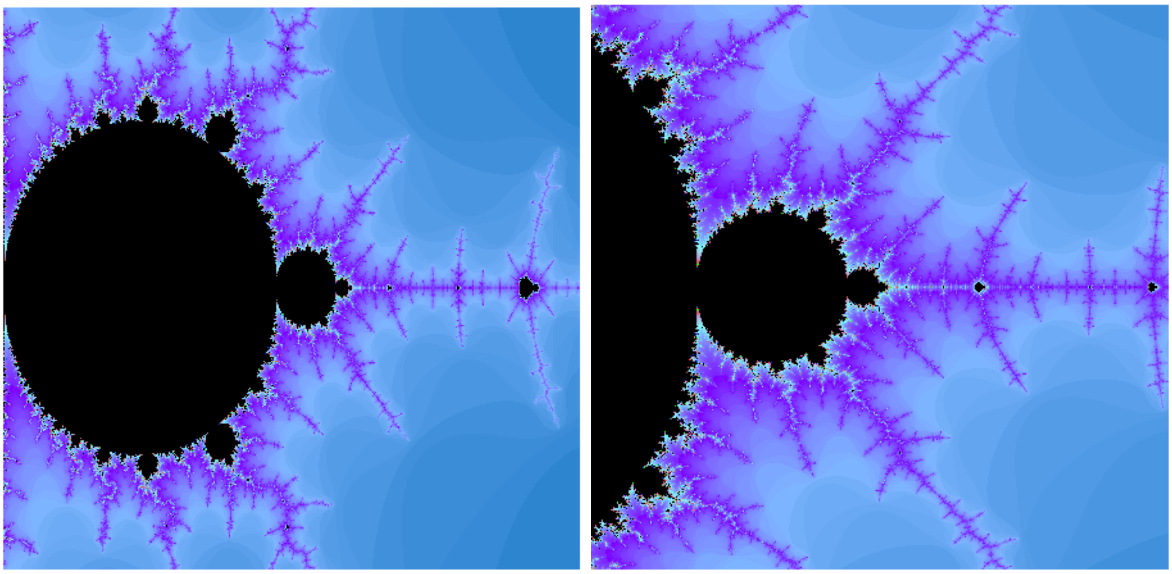

Figure 5.10: The Mandelbrot Set of the set of maps $f(z) = z^2 + c$ with $c \in I = [-1.25 - 0.1i, -1.5 + 0.1i]$ on the left, and $c \in I = [-1.35 - 0.05i, -1.45 + 0.05i]$ on the right [4].

In the following, we present a variety of examples of the Julia Set we can find in the quadratic family depending on the values of c we take. The first case considered is the function $f(z) = z^2 + c$ for an arbitrary value of c near to zero. Taking into account that $f(z) = z^2$ has an attracting fixed point on $z = 0$ and the Julia Set $\mathrm{J}(z^2)$ is the unit circle (given in figure 5.12), the circle clearly bounds both of the basins of attraction at $z = 0$ and $\{\infty\}$.

In our example, since $|c|$ is small a similar phenomenon occurs. As shown in figure 5.13, we still have an attracting fixed point near 0 for the values $c = 0.1$ and $c = 0.2$. We can also see the boundary of the Julia Set is a simple closed curve (which is far from being a smooth curve). But if we turn to the function at $c = 0.5$, figure 5.14 shows how the previous simple curve starts to degenerate into something else.

What we have observed in these pictures can be generalized in the following proposition:

Proposition 5.1 *If $c < |\frac{1}{4}|$, then the Julia Set of $Q_c(z)$ is a simple closed curve which contains no smooth arcs.*

Property 5.2 *If $c < |\frac{1}{4}|$, then the function $Q_c(z)$ has a repelling fixed point at $z_0 = \left(1 + \sqrt{1-4c}\right)/2$. Moreover, $Q_c'(z_0)$ is a complex number which is not pure imaginary, it means, z_0 does not lie in a smooth arc in $z(\theta)$.*

The last stated property allows us to extend Proposition 5.2 to other values of c bigger than $\frac{1}{4}$ as follows:

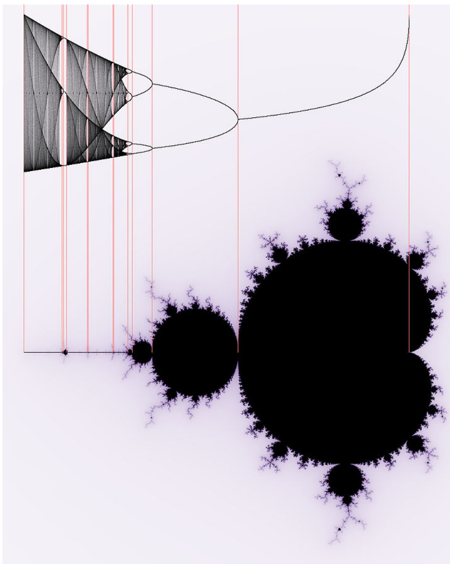

Figure 5.11: A correspondence between the Mandelbrot set and the bifurcation diagram for $Q_c(x)$.

Proposition 5.2 *If the quadratic function Q_c has an attracting point for some value of c, then the Julia Set of $Q_c(z)$ is a simple closed curve which contains no smooth arcs.*

Property 5.2 is satisfied for all the values of c inside the cardioid in the c-plane. Moreover, one can actually prove that the Julia Set for these Q_c is actually non-differenciable at every point on the simple closed curve.

Now we turn to the case of an attracting periodic point rather than a fixed point. To do so, it is studied the function $Q(z) = z^2 - 1$. Since $Q(0) = -1$, $Q^2(0) = 0$ and $Q'(0) = 0$, we can deduce $z = 0$ lie on an attracting periodic orbit of period 2 of the form $\{0, -1\}$.

To study the dynamics of $Q(z)$ on the real line, firstly, we calculate the fixed points of the function which are $z_1 = \frac{1-\sqrt{5}}{2}$ and $z_2 = \frac{1+\sqrt{5}}{2}$, both repelling fixed points as we can see iterating the function. The point z_1 corresponds to a dividing point between the basin of attraction of 0 and -1 (if we iterate $Q(z)$ with $z_0 = -0.5$ we obtain $Q(-0.5) = -0.75$ and so on, but if we iterate $Q(z)$ with $z_0 = -0.8$ we obtain $Q(-0.8) = -0.36$ and so on...). Also, from preposition we deduce there are two simple closed curves γ_0 and γ_1 in $J(Q_c)$ which surround 0 and -1 respectively. Both curves meet at the fixed point $\frac{1-\sqrt{5}}{2}$. There is much more to $J(Q_c)$ however. Unlike the situation for Q_c, the basin of attraction of 0 is not completely invariant. One preimage of the interior of γ_0 is clearly γ_1, but there must also be another surrounding the other preimage of 0, namely 1. That is, there is a third simple closed curve in $J(Q_c)$ surrounding 1 as well. Now both 1 and -1 must have a pair of distinct preimages,

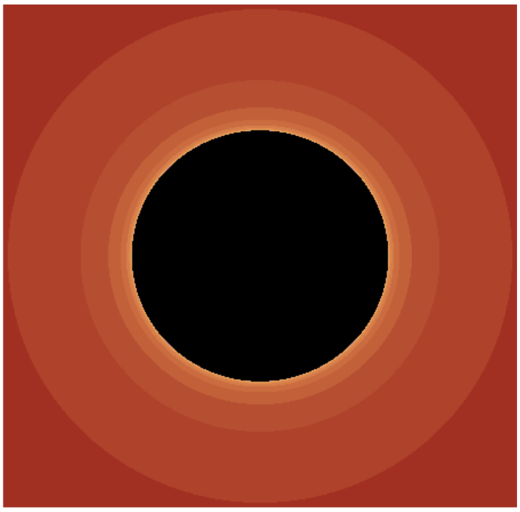

Figure 5.12: The Julia Set of $f(z) = z^2$ [4].

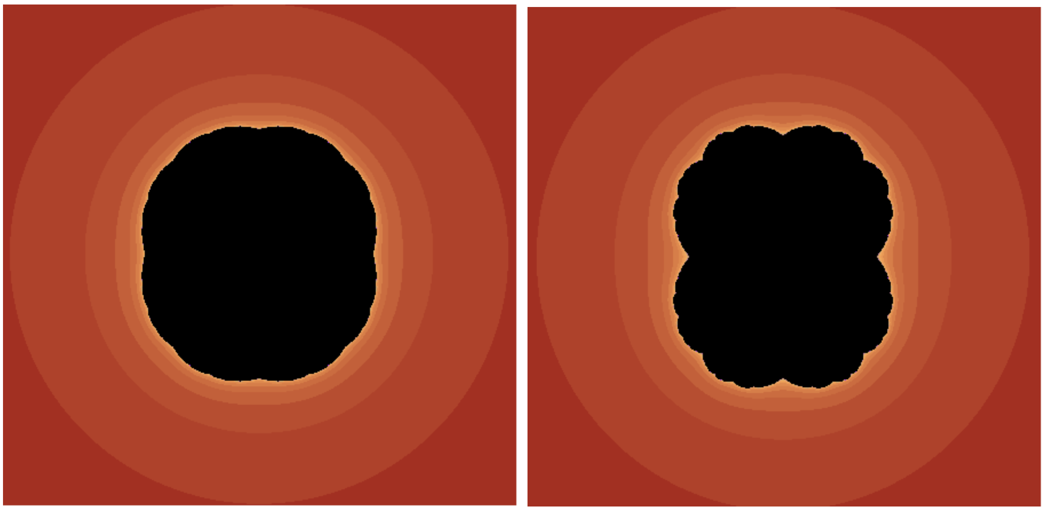

Figure 5.13: The Julia Set of $f(z) = z^2 + c$ with $c = 0.1$ and $c = 0.2$. [4].

each surrounded by a simple closed curve in $J(Q_c)$. Continuing in this fashion, we see that $J(Q_c)$ must contain infinitely many different simple closed curves. This fact is generalized in the following proposition:

Proposition 5.3 *Suppose a generic function $f(z)$ is a polynomial of degree 2. Then the stable set of P consists of either one, two, or infinitely many connected components.*

As a final example, we will consider the quadratic function $Q_c(z) = z^2 + c$ with $c \in \mathbb{R}$. Since $Q_c'(0) = 0$, $z = 0$ is a critical point of the function. Moreover, it is eventually periodic since $Q_c^3(0)$ is a repelling fixed point.

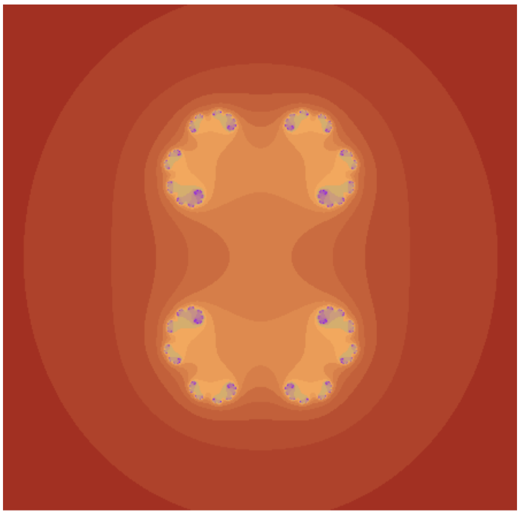

Figure 5.14: The Julia Set of $f(z) = z^2 + c$ with $c = 0.5$. [4].

In a more general way, it is satisfied that the repelling periodic points of Q_c contained in the interval $[-p, p]$ (where $-p = Q_c^3(0)$) is dense on $\mathbb{C}$. This interval has the important property that it is contained in the Julia Set, and since it is invariant, all the preimages of the interval also lie in the Julia Set. Indeed, The Julia Set $J(Q_c)$ is the closure of this set of intervals.

Now consider we have a second repelling fixed point at point q, it is satisfied that $[-q, q] \in J(Q_c)$. Since $c \in [-q, q]$, the preimage of this interval consists of two intervals ($[-q, q]$ and a second interval located symmetrically from $z = 0$ but on the imaginary axis). This second interval can be calculated as the preimage of $[-q, c]$, therefore, we have the preimage of this two intervals consists on four curves that intersect $z = 0$ and $z = Q_c^{-1}(0)$. In general, we deduce that $Q_c^{-n}([-q, q])$ consists of 2^n disjoint curving linear segments and the Julia Set $J(Q_c)$ therefore is formed by the closure of this set of preimages.

This structure given by

$$\bigcup_{n=0}^{\infty} Q_c^{-n}[-q, q]$$

is called a dentrite, which is depicted in figure 5.15. In this figure we show an example of a dentrite given by the function $Q(z) = z^2 + i$, but of course there are other values of c we could use to obtain a similar behaviour of the Julia Set. For example, $Q(z) = z^2 - 1 + i$ or $Q(z) = z^2 - i$.

A conclusion from the study of the Feigenbaum attractor is that depending on the value of c we consider we can obtain a vast array of different phenomena, from the ones obtained for

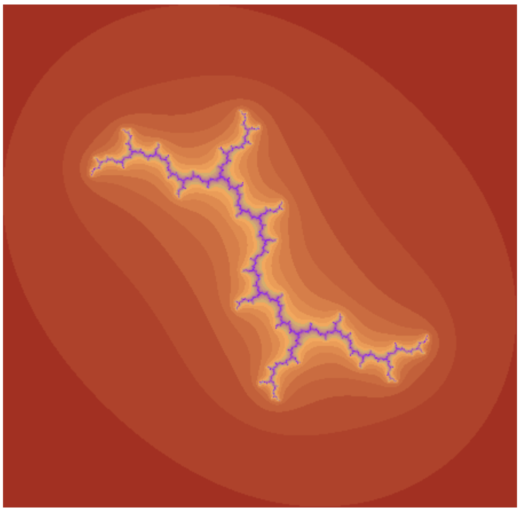

Figure 5.15: The Julia Set of $f(z) = z^2 + i$. [4].

the real case to the ones obtained for the complex case. The resulting locus in the parameter plane obtained by The Mandelbrot Set is still a subject of much contemporary research.

Chapter 6

The Lorenz Attractor

6.1 The Lorenz Equations

In this chapter, we will focus on the study of one of the best-known attractors so far, the Lorenz attractor. As mentioned in the introduction, the meteorologist E. Lorenz was the first person to relate the phenomenon of convection in the earth's atmosphere to the concept of attractors.

To find out more about them, he defined a continuous dynamical system governed by the following differential equations.

$$\left.\begin{array}{rcl} \dot{x} & = & \sigma(y-x) \\ \dot{y} & = & rx - y - xz \\ \dot{z} & = & xy - bz \end{array}\right\} \tag{6.1}$$

in which σ, r and b are positive parameters. Even though the use of the system (6.1) is highly questionable as a model for convection in the atmosphere, it allowed us to glimpse different behaviours of the system depending on the values we take as initial conditions. To visualize this phenomenon, some simulations of the Lorenz attractor will be displayed taking the same values for σ, r and b, but different initial values.

6.1.1 Simulations of the Lorenz Attractor

In this section, we carry out different simulations of the Lorentz attractor in order to see the different shapes that the Lorenz attractor can take. For this purpose, we carry on some numerical experiments considering the values $\sigma = 10$, $b = \frac{8}{3}$ and some positive value r (we will take several ones to make different simulations of the system). For now, let $r = 28$. The following code in figure 6.1 (written entirely in MATLAB) allow us to simulate the Lorenz attractor for an arbitrary initial condition.

As an example, four simulations are given in figure 6.2 considering the same values of the parameters σ, r and b but different initial conditions. Now, let $r = 15$ and $r = 128$, we can observe in figure 6.3 that the lower the value of r is, the more incomplete is the graphic. In the same way, the greater the value of r is, the fuller the graph looks. This curious shape of the Lorenz attractor is usually named as "the butterfly's Lorenz" since it seems to have a similar form of a butterfly.

```
function [x,y,z] = lorenz(rho, sigma, beta, initV, T, eps)
% LORENZ Function generates the lorenz attractor of the prescribed values
% of parameters rho, sigma, beta
%
%   [X,Y,Z] = LORENZ(RHO,SIGMA,BETA,INITV,T,EPS)
%       X, Y, Z - output vectors of the strange attactor trajectories
%       RHO     - Rayleigh number
%       SIGMA   - Prandtl number
%       BETA    - parameter
%       INITV   - initial point
%       T       - time interval
%       EPS     - ode solver precision
if nargin<3
  error('MATLAB:lorenz:NotEnoughInputs','Not enough input arguments.');
end
if nargin<4
  eps = 0.000001;
  T = [0 25];
  initV = [0 0.22 0.5];
end
options = odeset('RelTol',eps,'AbsTol',[eps eps eps/10]);
[T,X] = ode45(@(T,X) F(T, X, sigma, rho, beta), T, initV, options);
plot3(X(:,1),X(:,2),X(:,3));
axis equal;
grid;
title('Lorenz attractor');
xlabel('X'); ylabel('Y'); zlabel('Z');
x = X(:,1);
y = X(:,2);
z = X(:,3);
return
end
function dx = F(T, X, sigma, rho, beta)
    dx = zeros(3,1);
    dx(1) = sigma*(X(2) - X(1));
    dx(2) = X(1)*(rho - X(3)) - X(2);
    dx(3) = X(1)*X(2) - beta*X(3);
    return
end
```

Figure 6.1: MATLAB Code of the simulation of the Lorenz Attractor.

To execute this program, we need to put as input arguments the values of ρ, σ and b we want to consider to do the graphic of the Lorenz attractor. In case the program does not recognize the inputs as correct values, it will print “Not enough input arguments”. To be able to solve the ordinary differential equations given by (6.1), it is used the Runge-Kutta of 4th/5th order method. We take a sufficiently small error “eps=0.000001” to obtain a good approximation of the graph of the Lorentz attractor. Also, we take as initial value $(0, 0.22, 0.5)$ since our purpose is to build a numerical approximation of a solution which starts in a neighborhood of the unstable equilibrium solution $(x_0, y_0, z_0) = (0, 0, 0)$ (let us call this neighborhood U). Since we start in U, the orbit follows the unstable manifold $W^U(0)$ as near as possible.

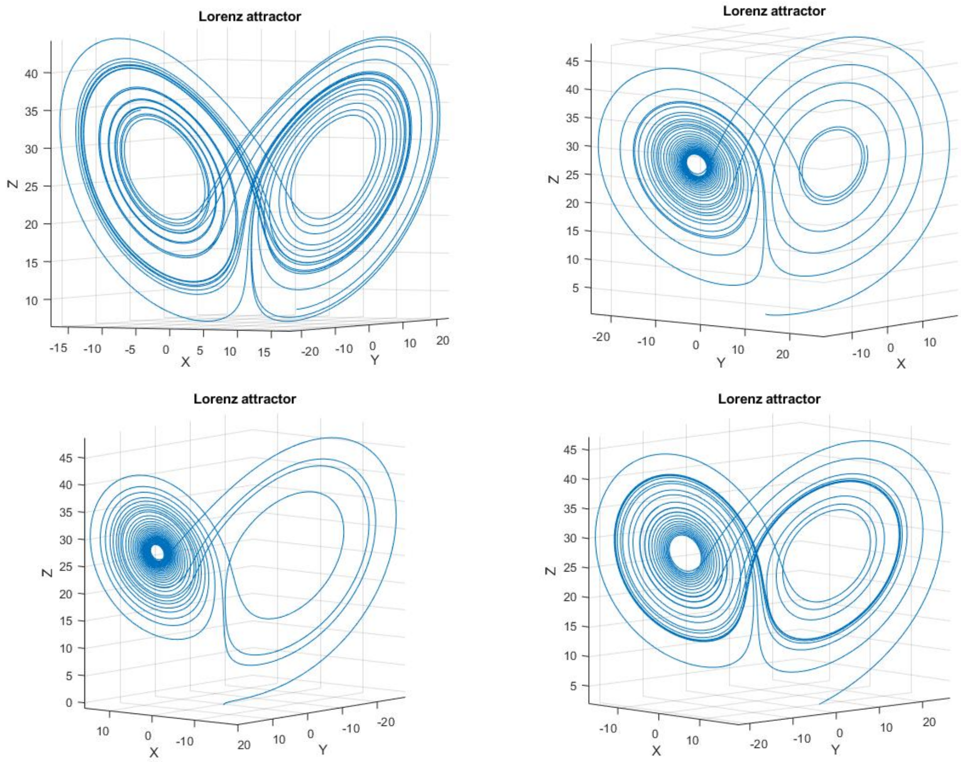

Figure 6.2: Simulation of the Lorenz Attractor using $r = 28, \sigma = 10$ and $b = 8/3$ considering as initial value $(0, 1, 1.05)$ up, on the left; $(0, 0.22, 0.5)$ up, on the right; $(0, 0.05, 0.05)$ down, on the left; and $(0, 0.10, 0.15)$ down, on the right.

As stated above, it can be visualized different behaviours of the attractor only by changing the initial conditions. This result, intriguing and fascinating at the same time, paved the way to the study of what it is called nowadays the chaos theory. However, this is not the only conclusion to be drawn from the simulations. On the one side, we can see the orbit is not closed. On the other side, the orbit does not represent a transition stage to well-known regular behaviour, instead of it, the orbit continues describing loops on the left and on the right without apparent regularity in the number of loops. Moreover, if we do different simulations, we can see the simulation is roughly the same picture. This suggests there exists an attractor with a dimension bigger than two, which, in the perspective of the equilibrium point x_0, has a complicated topological structure. This complicated structure is known as "strange attractors".

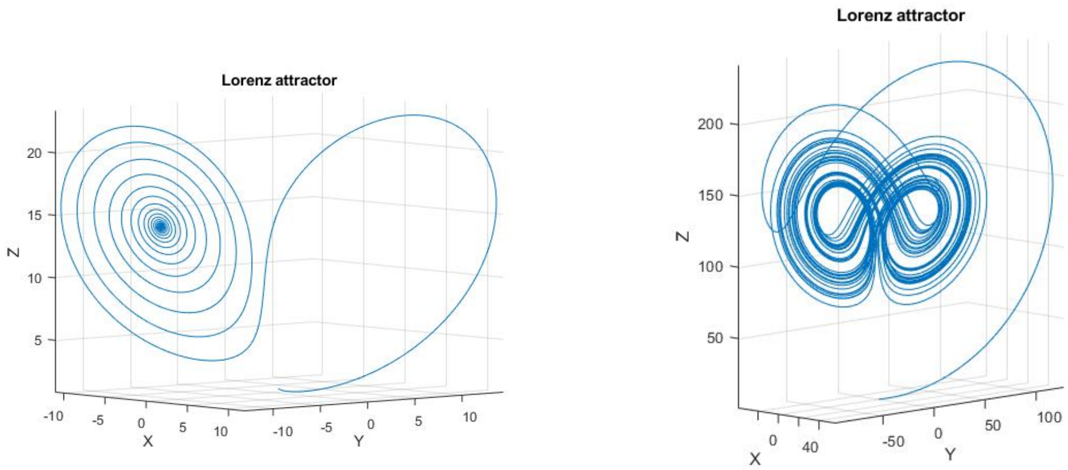

Figure 6.3: Simulation of the Lorenz Attractor using the values $\sigma = 10$, $b = \frac{8}{3}$ and two different values of r ($r = 15$ on the left, $r = 128$ on the right).

6.1.2 Critical Points and Equilibrium Solutions

In order to do a major study of the behaviour of the system, it is obtained the different equilibrium solutions of the system (6.1) depending on the values of r we consider.

In the case we take $0 < r < 1$, there is only one critical point in $(x, y, z) = (0, 0, 0)$ which is asymptotically stable. But, if we take $|r| = 1$, one eigenvalue becomes zero and hence a bifurcation occurs. This means, new critical points appear for $|r| > 1$. In particular, for $|r| > 1$ we have three critical points in $x_0 = (0, 0, 0)$, $x_1 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $x_2 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$.

To study these critical points in more depth, firstly, let me see what is the behaviour of the system at the origin for any value of r. To do so, we define the vector function of the Lorenz system as $F(x, y, z) = (\dot{x}, \dot{y}, \dot{z})$. The critical points of the system are given by the solutions of the equation $F(x, y, z) = (0, 0, 0)$. Hence, if we take the point $(x, y, z) = (0, 0, 0)$ and we substitute it on the function $F(x, y, z) = (\sigma(y - x), rx - y - xz, xy - bz)$, we deduce $F(0, 0, 0) = (0, 0, 0)$ and the origin is a critical point.

To be able to characterize this point, we calculate the Jacobian matrix of the function F,

$$J_F = \begin{pmatrix} -\sigma & \sigma & 0 \\ (r-z) & -1 & -x \\ y & x & -b \end{pmatrix},$$

which allow us to obtain the following system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (r-z) & -1 & -x \\ y & x & -b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Evaluating this system of equations at the origin, we are able to characterise this point obtaining the eigenvalues and eigenvectors of the system. To do so, we solve the equality $|J_F(0,0,0) - \lambda I| = 0$ as follows,

$$\begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} = 0 \quad \rightarrow \quad \big((-\sigma-\lambda)(-1-\lambda) - \sigma r\big)(-b-\lambda) = 0.$$

The equation obtained provide us with two new easier equations to obtain the eigenvalues. The first one is $(-b-\lambda) = 0$, from which we obtain $\lambda_1 = -b$. The second one is $(-\sigma-\lambda)(-1-\lambda) - \sigma r = 0$, from which we obtain λ_2 and λ_3,

$$\lambda_2 = \frac{1}{2}\left(-\sqrt{\sigma^2 + (4r-2)\sigma + 1} - \sigma - 1\right),$$

$$\lambda_3 = \frac{1}{2}\left(+\sqrt{\sigma^2 + (4r-2)\sigma + 1} - \sigma - 1\right).$$

The value of λ_1 is always negative since the parameter b is positive (we defined it at the start of the chapter). Also, the value of λ_2 is always negative since $-\sqrt{\sigma^2 + (4r-2)\sigma + 1}$ and $-\sigma$ are always negative values (we defined σ as a positive value also at the start of the chapter). Hence, we have that for any value of r, λ_1 and λ_2 will be negative. Let me study what is happening with the third eigenvalue.

We know λ_3 will be positive if $\sqrt{\sigma^2 + (4r-2)\sigma + 1} > \sigma + 1$. To deduce for which values of σ and r the inequality is satisfied, we solve it as follows,

$$
\begin{aligned}
\frac{1}{2}\left(+\sqrt{\sigma^2+(4r-2)\sigma+1}-\sigma-1\right)>0 &\iff \sqrt{\sigma^2+(4r-2)\sigma+1}-\sigma-1>0\\
&\iff \sqrt{\sigma^2+(4r-2)\sigma+1}>1+\sigma\\
&\iff \sigma^2+(4r-2)\sigma+1>(1+\sigma)^2\\
&\iff 1+\sigma^2-2\sigma+4\sigma r>1+\sigma^2+2\sigma\\
&\iff 4\sigma r>4\sigma\\
&\iff r>1,\ \sigma\neq 0.
\end{aligned}
$$

Therefore, we can say λ_3 will be positive if $r > 1$. On the other side, we know λ_3 will be negative if $\sqrt{\sigma^2+(4r-2)\sigma+1} < \sigma+1$. Therefore, we can say λ_3 will be negative for $r < 1$.

Remark. The previous study has been made assuming the value $\sigma^2 + (4r-2)\sigma + 1$ is positive, and hence we have real eigenvalues. In the case this value is negative, we would have two complex eigenvalues with real part $-\sigma - 1 < 0$, which means we would have two eigenvalues with negative real part. We know in this case we would have also an attractor.

This leads to the following preposition.

Proposition 6.1 *The critical point $x_0 = (0,0,0)$ will be an attractor in two cases. First case, if $r < 1$ and $\sigma \neq 0$. Second case, if $\sigma^2 + (4r-2)\sigma + 1$ is negative.*

Remark. We do have an attractor in the origin when we have three negative eigenvalues (which means λ_3 must be negative), otherwise, the origin cannot be an attractor.

In particular, if we take the values of $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$ we have been considering through the chapter and we substitute them on the eigenvalues, we obtain the following values: $\lambda_1 = -\frac{8}{3} \approx -2.66$, $\lambda_2 \approx -22.82$ and $\lambda_3 \approx 11.82$. Since we have λ_3 is positive, we have a saddle-point. Moreover, if we calculate the eigenspace (the set of all eigenvectors) associated to each eigenvalue we obtain:

$$S_{\lambda_1} = \{(0,0,z), z \in \mathbb{R}\},\ S_{\lambda_2} = \{(0.77y, y, 0), y \in \mathbb{R}\},\ S_{\lambda_3} = \{(0.46y, y, 0), y \in \mathbb{R}\}.$$

Hence, the following property is satisfied.

Property 6.1 *For $b = \frac{8}{3}, \rho = 28$ and $\sigma = 13$, the eigenvalues of the Jacobian matrix at the critical point $(0,0,0)$ are real, they have multiplicity equal to one and also opposite signs.*

Therefore, we can conclude the critical point $(0,0,0)$ *is a saddle point with a one-dimensional unstable manifold.*

Now we study what is happening in the critical points x_1 and x_2 we found for $r > 1$. In this case, both eigenvalues must satisfy the following equation,

$$\lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(\sigma + r) + 2\sigma b(r-1) = 0,$$

where we can see depending on the value of r the roots can be the following:

- First case, if $1 < r < \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$, there exist three roots of the equation. All of them have negative real parts, hence, there exist two unstable periodic solutions corresponding with two critical points.

- Second case, if $r = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$, two eigenvalues are purely imaginary and the previous two critical points vanish. A Hopf bifurcation occurs.

- Third case, if $r > \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$, we have two critical points. Each of them has one negative real eigenvalue and two eigenvalues with real part positive, i.e., they are unstable solutions.

For example, if we take the previous values of $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$, the Hopf bifurcation would occur at $\frac{\sigma(\sigma+b+3)}{\sigma-b-1} = 24.74$. To visualize the behaviour of the attractor depending if we take a value of r in the first, in the second or in the third case, a simulation of the Lorenz attractor for $r = 0$, $r = 24.74$ and $r = 28$ is given at figure 6.4.

6.1.3 Properties

Property 6.2 *The Lorenz-equations have reflection symmetry.*

Proof. Given the solution (x, y, z), we know the Lorenz equations look as equation (5.1). Now, if we consider the solution $(-x, -y, z)$, the Lorenz equations looks as follows:

$$\left.\begin{array}{rcl} -\dot{x} & = & \sigma(x-y) \\ -\dot{y} & = & -rx + y + xz \\ \dot{z} & = & xy - bz \end{array}\right\}$$

And if we multiply the first and second equations by (-1) on the left and on the right side, we obtain:

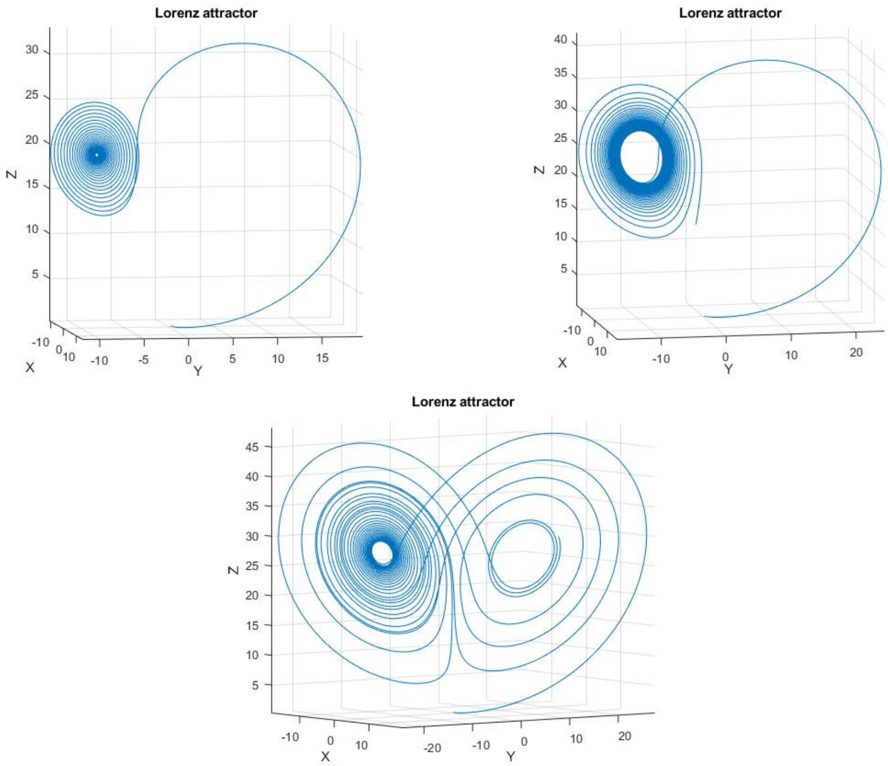

Figure 6.4: Simulation of the Lorenz Attractor using $r = 0$, $r = 24.74$ and $r = 28$ respectively.

$$\left.\begin{array}{rcl} \dot{x} & = & \sigma(y - x) \\ \dot{y} & = & rx - y - xz \\ \dot{z} & = & xy - bz \end{array}\right\}$$

which is of the same form as the Lorenz equations. □

As a conclusion, we obtain that given some particular solution (x_0, y_0, z_0), we know there exists another solution of the form $(-x_0, -y_0, z_0)$.

Property 6.3 *The set* $\{(x, y, z)/x = 0, y = 0, z \in \mathbb{R}\}$ *is an invariant set. Moreover, these solutions* (x, y, z) *tend to the equilibrium point* $(0, 0, 0)$ *for* $t \to 0$. *A sketch of the proof is given in* $\{$*[13], page 206*$\}$.

It is possible to obtain bounded, invariant sets in which the solutions are contained for a number of iterations. For this purpose, we consider the following Lyapunov function:

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2.$$

The orbital derivative with respect to the system is given by:

$$L_t V = -2\sigma(rx^2 + y^2 + bz^2 - 2brz).$$

It is satisfied that $L_t V \geq 0$ in the domain of

$$D = \{(x, y, z) \in \mathbb{R}^3, rx^2 + y^2 + b(z - 2r)^2 \leq 4br^2.\}$$

If we denote by M the maximum value of V in the ellipsoid bounded by $rx^2+y^2+b(z-2r)^2 = 4br^2$, we define another ellipsoid denoted by

$$E = \{(x, y, z) \in \mathbb{R}^3, V(x, y, z) \leq M + \epsilon.\}$$

Let P be a point of the phase-space outside of E, then, it is also outside D so that $L_t V(P) \leq -\sigma < 0$, given $\sigma > 0$. That means, the function V must decrease along the orbit which starts in P, and this means, after some iterations the orbit must enter in E and it will never leaves it.

Therefore, we can conclude for the case $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$ the flow enters an ellipsoidal domain E which contains three unstable equilibrium solutions (see figure 6.5). Moreover, from the boundness of E and the shrinking of each volume-element in the flow, the $\omega-$limit set may contain irregular orbits which give rise to the name "strange attractor".

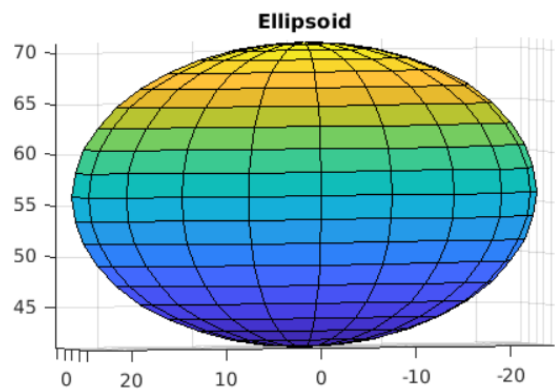

Figure 6.5: A simulation of the ellipsoid bounded by $rx^2 + y^2 + b(z - 2r)^2 = 4br^2$ with $\sigma = 10$, $b = \frac{8}{3}$ and $r = 28$.

6.2 Proof of existence of a strange attractor

Once we have seen some of the most important properties of the Lorenz attractor, we prove it indeed exists. To do so, we base our study on the geometric model used to describe the dynamics of the flow introduced by Guckenheimer and widely explained by Warwick Tucker at [11].

At the beginning of the section we define the main concepts we will use throughout the demonstration. Firstly we define what is the Poincaré map and the problem involved in its study and secondly we perform a change of variable that will transform the Lorenz equations into a normal form suitable for further study. Our aim with this study is to find the evolution of the trajectories analytically. Once we have done this, our next goal is to prove an attracting set N is in fact the basin of attraction of the Lorenz attractor.

In what follows, we base our work on the proof given in ([11], pages 1199-1202). All the codes and data needed to carry out the proof are given in

http://www2.math.uu.se/~warwick/main/thesis_2.1.html

Hence, to do so, we start considering a Poincaré section Σ (see figure 6.6). The flow of (6.1) can be defined as a Poincaré map Q acting on the section $\Sigma \in \{z = r - 1\}$, except for the line $\Gamma = \Sigma \cap W^S(0)$, i.e., the intersection between Σ and the stable manifold of the origin. By the definition of $W^S(0)$ we know all points inside this manifold tend to $z = 0$ (in the previous section a extended study of the origin has been made) and consequently, these points never come back to the Poincaré section Σ. Therefore, if we take some point $p \in \Gamma$, it will lead to a serious problem in numerical approximation. To overcome this problematic, we introduce a local change of coordinates of the form $t = s + \phi(s)$.

Our main objective by applying the above change of variable is to transform the Lorenz equations given in the Jordan normal form $\dot{x} = Ax + F(x)$ into a normal form which is virtually linear in a small cube centered at $z = 0$. Hence, inside this cube, we are able to estimate the evolution of trajectories analytically, and thereby we avoid the problem of having to use computers in regions where the flow times are unbounded.

Afterwards, we use a linearization method introduced by Poincaré based on an analytic change of coordinates to obtain the desired estimates of these trajectories. All of these steps are summarized in the following preposition, which provides us with the estimates we are looking for.

Proposition 6.2 *There exists a close to identity change of variables $t = s + \phi(s)$ satisfying*

$$\|\phi\|_r \leq \frac{r^2}{2}, \; r \leq 1,$$

such that the Lorenz equations in Jordan normal form $\dot{t} = At + F(t)$ are transformed into the normal form $\dot{s} = As + G(s)$, where $G(s) \in O^{10}(s_1) \cup O^{10}(s_2, s_3)$ and also satisfies,

$$\|G\|_r \leq 7 \cdot 10^{-9} \cdot \frac{r^{20}}{1 - 3r}, \; r < \frac{1}{3},$$

Remark: Some clarifications of the notation used in the above proposition are given below:

- The vector notation $s = (s_1, s_2, s_3)$ and the multi-index notation $a_n s^n = a_{n_1,n_2,n_3} s_1^{n_1} s_2^{n_2} s_3^{n_3}$.
- We suppose $s^n \in O^{10}(s_1) \cup O^{10}(s_2, s_3)$ if $n_1 \geq 10$ and $n_2 + n_3 \geq 10$.
- All the prove will be done inside a neighbourhood $V \subset \mathbb{C}$ of the point $z = 0$.
- We consider the following norms,

$$|s| = \max \{|S_i|, \quad i = 1, 2, 3\}$$

$$\|f\|_r = \max \{|f_i|, |s_i| \leq r\}.$$

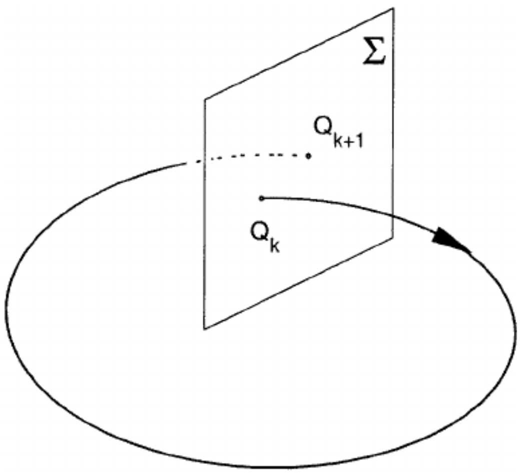

Figure 6.6: The Poincaré map acting on Σ.

Since the change of variable used and its inverse are analytic, as well as the function G obtained before, we can use the Cauchy-Riemann estimates to obtain information about its derivatives, and consequently, to estimate the exit of any trajectory and the entering of any tangent vector into the cube. To do this, we split the 3-space of natural numbers $\mathbb{N}^3$ into two disjoint sets U and V such that

$$V = \{(n_1, n_2, n_3) \in \mathbb{N}^3, n_1 < 10 \ \text{ or } \ n_2 + n_3 < 10\}$$

and U is the complementary set of V. Also, considering a function f such that $f(s) = \sum_n \alpha_n s^n$, we have the images of f by U and V are defined as follows,

$$\{f(s)\}_u = \sum_{n \in U} \alpha_n s^n,$$

$$\{f(s)\}_v = \sum_{n \in V} \alpha_n s^n.$$

If we apply now the linear operator L_A to the function $\phi(s)$ we obtain,

$$L_A \phi(s) = D\phi(s) A s - A\phi(s)$$

which leaves the spaces of homogeneous vector-valued polynomials of any degree invariant. In particular, taking it component to component we have,

$$L_{\Delta,i}\left(s^n\right) = (n\lambda - \lambda_i) \cdot s^n, \ i = 1, 2, 3$$

and if we substitute it into the following equation

$$L_A \phi(s) = F(s + \phi(s)) - D\phi(s) G(s) - G(s),$$

we obtain the two equalities

$$L_{A,i} \phi_i(s) = \{F_i(s + \phi(s))\}_V, \quad G_i(s) = \{F_i(s + \phi(s))\}_U - \sum_{j=1}^{3} \frac{\partial \phi_i}{\partial s_j}(s) G_{j(s)}, \ i = 1, 2, 3.$$

To solve the equation $L_{A,i}\phi_i(s) = \{F_i(s + \phi(s))\}_V$ we will make use of the following series

$$\phi_i(s) = \sum_{inl=2}^{\infty} a_{i,n} s^n, \ i = 1, 2, 3.$$

Remark: to avoid high-order resonances, any s^n appearing in the power series of ϕ must satisfy that $n \in V$. This property allows us to prove the following lemma, which gives the existence of a formal series of ϕ with the aid of some computer calculations.

Lemma 6.1 *For any $n \in V$ with $|n| \geq 2$, the divisors $n\,\lambda - \lambda_i, \ i = 1, 2, 3$ are bounded away* *from zero. Furthermore, for $|n| \geq 58$, there exists a sharp lower bound on the modulus of these divisors:*

$$|n\lambda - \lambda_i| \geq |9\lambda_1 + (|n| + 9)\lambda_3 - \lambda_i|\,,\ i = 1, 2, 3.$$

To prove the divisors are bounded away from zero, it is necessary to compute it using the computer, but we can do an analytic prove to demonstrate the sharp lower bound on the higher-order divisors. Our goal is to prove that the power series defined before is in fact convergent. To do so, the problem can be reduced to proving the convergence of a single variable power series of the form $\psi(r) = \sum_{k=2}^{\infty} C_k r^k$ satisfying $\|\phi\|_r \leq \psi(r)$. The coefficients of ϕ are given by the following recursive scheme:

$$c_k r^k = \frac{5}{\Omega(k)} \left[\left(r + \sum_{i=2}^{k-1} c_i r^i \right)^2 \right]_k, \ k \geq 2.$$

where we denote $\left[\sum \alpha_n s^n\right]_k = \alpha_k S^k$ and $\Omega(k) = \min\{|\lambda n - \lambda_i|\,, |n| = k, \quad n \in V, i = 1, 2, 3\}$.

Due to the recursiveness, the first coefficients have a large effect on the radius of convergence. In order to enlarge the radius, we estimate the 186,576 first coefficients $a_{i,n}$ of ϕ and we set $c_k = \sum_{|n|=k} \max_{i=1,2,3} |a_{i,n}|$ for $k = 2, ..., 70$ before using the recursive scheme. This gives a very good bound on the bound of $\|\phi\|_r$. Using similar techniques in $G_i(s) = \{F_i(s + \phi(s))\}_U - \sum_{j=1}^{3} \frac{\partial \phi_i}{\partial s_j}(s) G_{j(s)},\ i = 1, 2, 3.$ we can also get a bound on the normal form G (in a simpler way since no divisors appear in this equation).

Considering the return plane $\{z = \rho - 1\}$ where we defined previously our Poincaré section, we define a region N composed by two disjoint components N^- and N^+, each one formed by 350 adjacent rectangles (denoted by $N_i^{\pm}$) which belongs to the return plane as follows

$$N = N^- \cup N^+ = \left(\bigcup_{i=1}^{350} N_i^- \right) \cup \left(\bigcup_{i=1}^{3} N_i^{+0} \right).$$

Our main goal in this section will be to prove this region N is in fact the basin of attraction of the Lorenz attractor. As we have seen in the previous section, one of the most important properties of the Lorenz attractor is that it has reflection symmetry. This property is preserved on the two components of the region N, i.e., $N^+ = S(N^-)$, where S is the symmetry function. This property allows us to simplify our demonstration, since we only need to do the prove for one of the components of N.

Therefore, we consider each of the rectangles N_i of the component N (the sign is omitted since both of the components can be taken) and we compute a pseudo-path which strictly contains the flow of N_i. To obtain this pseudo-path, we need to introduce several intermediate return planes Π^m which are either $xy-$planes if it is satisfied the inequality $|\dot{x}| \leq |\dot{z}|$ or $yz-$planes if $|\dot{z}| \leq |\dot{x}|$ respectively. We start computing the pseudo-path of the rectangle N_1 introducing the first plane Π^1, to do so, the rectangle N_i is flowed to the first plane Π^1 by using the Euler Method for a sufficiently large error. In Π^1, we take the rectangular hull of the largest image of N_1, giving us a new starting rectangle R^1. This rectangle is then flowed to the second plane Π^2 and so on until we return to the Poincaré section Σ as we can see in figure 6.6. In the case that we have a rectangle R^m too large to continue with the process, we need to partition it into smaller rectangles, to be able to treat them separately.

In this section, rectangles N_i are being used to simplify the computations needed to sketch the prove. When a rectangle N_i is flowed between two intermediate planes Π^m and Π^{m+1}, usually the corners of the new rectangle R^m are the ones that yield the largest rectangular hull $R^{m+1} \subset \Pi^{m+1}$. Hence, we can reduce the error analysis to small pieces of R^m which will significantly reduce local errors. However, since computers work with finite precision, to be sure this property holds we need to test it in every step of the method.

We now move on to the cone field, to define this field we associate each rectangle N_i with a cone c_i. Each cone is given by two angles α_i and α_j whose boundary vectors are u_i and v_j respect to the x-axis (see figure 6.7).

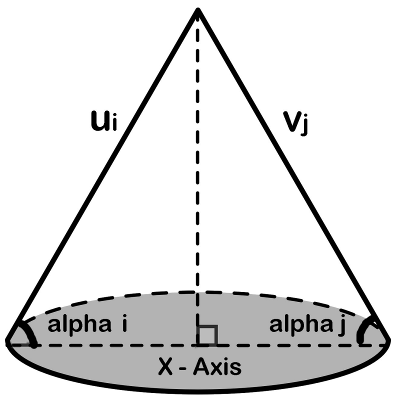

Figure 6.7: Cone C_i associated to N_i.

Then, taking the angle $\theta = \frac{\pi}{18}$ given by $\theta = \alpha_i - \alpha_j$ we use similar techniques as the ones used in the previous paragraph. Once a rectangle R^m has been flowed from the return plane Π^m to Π^{m+1}, we obtain a box containing the path of the rectangle. Not only that, the method also provides us with both upper and lower bounds on the flow time involved. Also, when we solve the nine equations governing the partial derivatives of the flow, we obtain rigorous bounds on the evolution of the tangent vectors flowing through the box. If we translate the flowed vectors onto the plane Π^{m+1} and we select some pair of vectors u_{m+1} and v_{m+1} making the largest angle θ_{m+1}, we ensure that the resulting cone contains all images of tangent vectors from the initial cone. Therefore, at the pseudo-return of N_i (which consists on many overlapping rectangles $Q_{i,j}$ for $j = 1, ..., k(i)$ such that $R(N_i) \subset \bigcup_{i,j} Q_{i,j}$), each rectangle $Q_{i,j}$ is this associated to a cone represented in the same way as before for $j = 1, ..., k(i)$.

Now taking the widest pair of vectors u_m and v_m at each intermediate plane Π^m and considering that we have $\theta^{m+1} \leq \theta^m$, then, the minimal expansion of each cone (denoted by ϵ_m) is the smallest growth factor of the images of u_m and v_m. But, if we have $\theta^m \leq \theta^{m+1}$, we must adjust this estimate by a factor which is quadratically close to unity in θ^{m+1}. Hence, at the return each rectangle $Q_{i,j}$ is associated with an expansion estimate of the form $\epsilon_{i,j} = \prod_{m=0}^{n} \epsilon_{i,j}^{m}$. Also, an estimate of all the vectors of the cone associated with N_i are given by $\epsilon_i = min_j \, \epsilon_{i,j}$.

Using this method, we can avoid the errors associated with computer's floating point calculations using a high-dimensional analogue of interval arithmetic. This means, each object a we consider (it could be a rectangle or a tangent vector for example) subjected to computation is associated with a maximal absolute error Δ_a that can be represented as a Cartesian product of the form

$$a \pm \Delta_a = [a_1 - \Delta_{a_1}, a_1 + \Delta_{a_1}] \times \cdots \times [a_n - \Delta_{a_n}, a_n + \Delta_{a_n}].$$

As we said before, when an object is flowed from an intermediate plane to another, we obtain upper and lower bounds of this object. In particular, we can compute upper and lower bounds of $a_i + \Delta_{a_i}$ and $a_i - \Delta_{a_i}$ for $i = 1, ..., n$ obtaining a new Cartesian product which strictly contains $a \pm \Delta_a$. Moreover, this gives us an error "sufficiently large" to accept any rounding error cause by the computation (as we assumed in the beginning of the prove).

This process is valid as far as we do not flow close to a fixed point, since in this case the local Poincare maps are well defined diffeomorphisms, and the computer can handle all calculations involved. If some rectangle approaches a fixed point, the process would be interrupted.

Therefore, we have obtained that the program previously defined verifies that $\bigcup_{j=1}^{k(i)}$ is strictly contained in N_i for each i, and we conclude $Q(N \setminus \Gamma) \subset int(N)$, as desired. It also proves the existence of a forward invariant cone field, i.e., $\forall x \in N$, $DQ(x) \cdot \mathbb{C}(x) \in \mathbb{C}(Q(x))$.

We have found regions in N which were contracted in all directions under Q. However, we prove that all tangent vectors within the cone field are eventually expanded under DQ. This means, we have been able to prove that given any pseudo-orbit $O(x_0) = \{x_0, x_1, ..., x_0\}$ (where $x_i = Q^i(x_0)$ the i-iteration of Q) we can divide it into disjoint intervals of the form $[x_0, ..., x_{k_0}]$, $[x_{k_0+1}, ..., x_{k_1}], ..., [x_{k_{n-1}+1}, ..., x_{k_n}]$ where all but the first piece accumulate an expansion factor greater than 2 (which leads to the transitivity of the function).

Chapter 7

Conclusion

- After reading this paper, it is important clear out that the field of dynamical systems is a fairly new branch of mathematics and there is still a long way to be explored. The aim of this paper is to introduce the concept of the attractor, and to show its great relevance in the study of dynamical systems. Although my aim was to show some of the most important properties discovered so far of easier attractors as the fixed points and most elaborated ones as the Lorenz attractor, there are still many unanswered questions about this subject.

- In the process of writing this paper I have discovered the exciting world of dynamical systems where practically any physical system whose state evolves over time can be modelled by a dynamical system (regardless of the area to which it belongs). In addition, the number of attractors that exist in nature is enormous, from natural events such as the swirl of water that appears when the bathtub is unclogged or the tides of the ocean to stranger ones discovered experimentally, all of them of a formidable complexity that is not possible to describe even on five papers like this. The great variety of attractors that exist, from the easiest to the most complex, can give rise to surprising results. And these attractors can occur not only in complex functions, a simple linear function can have a lot of different attractors.

- In my opinion I have reached all the goals I had in mind, but maybe I would have liked to do a deeper study on the strange attractors at the end of this dissertation. However, due to time constraints and the complexity of the subject we have at hand, it has not been possible for me to do so.

- The planning of the work made since the beginning has been successful, but maybe I should have started writting earlier to have been able to add this last section about strange attractors (anyways, an introduction to them has been presented with the Lorenz attractor).

- In a future work I would love to dive deeper on attractors. In this dissertation I was only able to show the different kind of attractive points there exist (together with a brief explanation of the Feigenabum attractor and the Lorenz attractor), but in the future I would love to know something more about periodic orbits who are also attractive (there is also a whole theory about this). I think there are many things that have remained unexplored and my aim would be to study them in future projects if I have the opportunity.

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