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A Sparse Bayesian Approach for Joint Feature Selection and Classifier Learning

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Abstract In this paper we present a new method for Joint Feature Selection and Classifier Learning (JFSCL) using a sparse Bayesian approach. These tasks are performed by optimizing a global loss function that includes a term associated with the empirical loss and another one representing a feature selection and regularization constraint on the parameters. To minimize this function we use a recently proposed technique, the Boosted Lasso algorithm, that follows the regularization path of the empirical risk associated with our loss function. We develop the algorithm for a well known nonparametrical classification method, the Relevance Vector Machine (RVM), and perform experiments using a synthetic data set and three databases from the UCI Machine Learning Repository. The results show that our method is able to select the relevant features, increasing in some cases the classification accuracy when feature selection is performed.

1 Introduction

Binary classification problems take as input a training data set $D = \{(\mathbf{x}^n, y_n)\}_{n=1,..N}$, where $\mathbf{x}^i \in \mathbb{R}^d$ is a sample (being d the subspace dimensionality) and y_i is an integer value representing its corresponding label. The goal is to learn a classifier that predicts the correct label of any unseen sample $\mathbf{x} \in \mathbb{R}^d$. Usually the data dimensionality d is large, which increases the difficulty of the classifier parameters estimation. This phenomenon is known as *the curse of dimensionality* [1,2], and it has been shown that the number of samples needed to learn a reliable classifier increases exponentially with the dimensionality d. To overcome this drawback, two different approaches are followed in the current machine learning research: (i) to use robust classification rules, such as Support Vector Machines [3], boosting variants [4] or Bayesian logistic regression approaches [5], and (ii) to perform a previous preprocessing step on the input data, obtaining the most informative feature subset for the classification task [6,7]. When only a subset of features from the original data are preserved, the process is known as *feature selection* [8], while when the original features are combined by applying a learned mapping to a reduced space the process is known as *feature extraction* [9–11].

Feature selection has been shown to improve generalization when many irrelevant features are present. According to Guyon and Elisseeff work [12], feature selection approaches are divided into filters, wrappers and embedded approaches. Most common approaches are filters, which act as a preprocessing step independently of the final classifier. In contrast, wrappers take the classifier into account as a black box. Finally, embedded approaches simultaneously determine features and classifier during the training process.

There are specifically tailored methods for feature selection in nonparametric classifier learning. Guyon et al. [13] proposed a feature ranking method for Support Vector Machines where the selection is performed by Recursive Feature Elimination (RFE). Mao [14] used a pruning strategy for the same Support Vector Machines framework. In the same context Chen et al.[15] used the Fisher ratio in a forward feature selection for constructing sparse kernels.

Hong and Mitchell [16] used the leave-on-out criteria in a backward selection algorithm applied to post-processing procedures for classification problems while Weston et al. [17] used the gradient descent method over an upper bound of the leave-one-out error for selecting the optimal subspace. On the other hand, Automatic Relevance Determination (ARD) [18] has been used to estimate scaling factors on features in different types of classifiers. In kernel classifiers such as Support Vector Machines a Bayesian method for model selection was proposed by Seeger [19], where a maximum a posteriori (MAP) criterion on the parameters is imposed using a variational approach. Nevertheless, there exist only a few embedded methods addressing the feature selection problem in connection with non-parametric classifiers up to now. An embedded approach for the quadratic 1-norm SVM was suggested by Zhu et al. [20], while Jebara and Jaakkola [21] developed a feature selection method as an extension to the so-called maximum entropy discrimination. Li at al [22] used a two stepwise forward procedure for the same compacting model problem. Recently, Krishnapuram et al. [23] developed a *joint classifier and feature optimization* method for kernel classifiers. The method shows promising results in high dimensional data sets. Nevertheless, the resulting kernel classifier model is learned by the Expectation Maximization algorithm and the computational complexity becomes unpractical for large training data sets.

In this paper we propose an embedded method to jointly perform feature selection and classification for a well known non-parametric classifier, the Relevance Vector Machine (RVM) [24]. Our method uses a Laplace prior, through an L_1 penalty or constraint, to promote sparsity on the RVM parameters as well as in the distribution of selected features. Tibshirani et al. [25] were the first to propose this prior to achieve sparse parameter estimates in the regression context. Since then, the use of constraints based on the absolute values of coefficients has been used to achieve sparseness in a variety of learning tasks, such as L_1-SVM or Logistic Regression Learning [5]. However, Laplace priors produce a loss function that is not differentiable, so that we can not use any gradient or higher order derivatives to minimize our loss function.

Recently, a new method for coordinate descent that does not rely on derivatives has been proposed for the optimization of general convex loss functions penalized through an L_1 constraint [26]. This algorithm, known as Boosted Lasso (BLasso) [27], can be easily extended to the case of a general loss penalized by a general convex function. Our main contribution in this work is the use of a BLasso approach to jointly perform the feature selection and the classifier training tasks.

This paper is organized as follows. In the next section we give a brief review of the main statistical tools used in this work: the Vector Machines classifiers, the Lasso problem and the BLasso algorithm, which yields to the proposed new approach of the RVM, the Laplacean-RVM (L-RVM). Section 3 describes the proposed Joint Feature Selection and Classifier Learning method (JFSCL). In section 4 we expose some experimental results for different purposes: (*i*) comparison of classical RVM approach with L-RVM, (*ii*) comparison of L-RVM with the proposed JFSCL, and (*iii*) comparison of JFSCL with some of the state-of-the-art feature selection filters. Moreover we also analyze the sensitivity of the proposed JFSCL with respect to the design parameters of the method. Finally section 5 concludes this work showing also some future lines of research.

2 Sparse Bayesian Learning using Laplace Priors

In this paper we present an embedded method to jointly perform the feature selection and classifier training tasks in a Bayesian framework. Although we will work on a specific classifier for this purpose (RVM), the proposed framework could be used for almost any non-parametric classifier. The election of the RVM classifier is justified by its performance in general learning problems as well as by its elegant adaptation to the sparse Bayesian learning framework. This classifier fares similar to the "best of-the-shelf" SVM classifier when considering its classification performance while overcomes SVM main drawbacks: it gets a real sparse vector selection, it has a lower computational cost and it is less prone to overfitting [24].

2.1 Vector Machines

Support Vector Machine (SVM) [28] is one of the most successful classifiers developed in the pattern recognition field. Briefly, considering a 2-classes classification problem, a SVM predicts the expected labels as

$$y(\mathbf{x}) = \chi_{(0,\infty)}(\sum_{n=1}^{N} w_n K(x, x_n) + w_0) \tag{1}$$

being $\chi_{(0,\infty)} : \mathbb{R} \rightarrow \mathbb{R}$ the characteristic function (defined to be identically 1 on $(0, \infty)$ and 0 elsewhere), K a kernel function, and $\mathbf{w} = (w_0, ..., w_N)$

the weights vector that should be learned to minimize the empirical classification error. The maximization of the margin criterion has been shown to be an appropriate property to reduce the generalization error, and some upper bounds have been defined in the literature [28]. Nevertheless, the method suffers from some limitations, such as:

- predictions are binary, and no probabilistic value can be derived from the classification of a sample.
- usually, the number of selected support vectors is small, but no specific prior is imposed on the solution to control the desired sparsity level.
- in general, the solution that minimizes the training error is not the solution with maximum margin, and thus this solution is not the one with expected minimum generalization error. A trade off parameter is needed to fix this optimal operation point. This process can only be solved by cross validating the training set, being the tuning process computationally costly.
- there is a variety of kernel functions K to use, but they must satisfy the Mercer's condition.

To overcome these drawbacks, Tipping [24] proposed the RVM classifier, where a Bayesian approach for learning the parameters $\mathbf{w}$ is followed. Briefly, he suggested the use of a set of hyperparameters associated to the generation of the weights $\mathbf{w}$, obtaining a sparse distribution on $\mathbf{w}$. The experimental results show a generalization capability close to the SVM, using an impressively few number of support vectors.

As a first step to our objective, we propose the use of a Laplace prior to promote sparsity on the Relevant Vector Machine parameters. Formally, and given the samples of a 2-classes classification problem, let us adopt a Bernoulli probabilistic distribution on the prediction of the RVM as follows

$$P(y=1|x) = \sigma(\sum_{n=1}^{N} w_n K(x, x_n) + w_0) \tag{2}$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is the sigmoid function, $\sigma(a) = 1/(1+\exp(-a))$, for all $a \in \mathbb{R}$. From the definition of a Bernoulli distribution we have $P(y=0|x) = 1 - P(y=1|x)$, and the negated log-likelihood estimator for the parameters set $\mathbf{w} = (w_0, ..., w_N)$ is

$$L(D, \mathbf{w}) = -\sum_{i=1}^{N} \sigma(\sum_{n=1}^{N} w_n K(x_i, x_n) + w_0)^{y_n} (1 - \sigma(\sum_{n=1}^{N} w_n K(x_i, x_n) + w_0)^{1-y_n}) \tag{3}$$

To promote sparsity on the selected vectors we can penalize the loss function $L(D, \mathbf{w})$ through a constraint such as $R(\mathbf{w}) = \|\mathbf{w}\|_1$, getting the following loss function

$$G(\mathbf{w}) = L(D, \mathbf{w}) + \lambda R(\mathbf{w}) \tag{4}$$

being λ a positive real value. In this formulation the estimate for $\mathbf{w}$ when using this model can be interpreted as a Bayesian posterior mode estimate when the prior on the parameters are independent double-exponential (Laplace) distributions [26].

There are two aspects of this new RVM approach to be studied in order to become a real alternative to the original one: (i) how to compute a feasible solution, and (ii) how this solution compares to the solution obtained by classical RVM.

Notice that this RVM approach corresponds to a Lasso problem, that is, a Loss Function with a penalty on the L1-norm of the parameters. In the next subsection we describe in detail the Lasso problem and include a brief survey on Lasso methods and their applications.

2.2 The LASSO Problem

Traditional approaches to learning methods seek to optimize a function associated to the empirical error rate. For example, most of the statistical procedures use likelihood or negated likelihood estimators to learn the parameters of the model. However, given that the training data is usually a finite set, learning algorithms could lead to an arbitrary complex model as a consequence of the training error minimization, obtaining in that way a classifier that presents high variance on unseen data and poor classification results on the testing data. To avoid this fact some regularization constraints on the parameter set can be jointly imposed during the training step. Thus, the learning method optimizes a loss function subject to some model complexity restriction.

Lasso is a regularization approach for parameter estimation originally proposed by Tibshirani [26] that shrinks a learned model through an L_1

penalty on the parameters set. Formally, in a statistical classification framework with parameters $\mathbf{w}$, the Lasso loss can be expressed as

$$G(\mathbf{w}, \lambda) = L(D, \mathbf{w}) + \lambda R(\mathbf{w}) \tag{5}$$

where $L(D, \mathbf{w})$ is the negated empirical error rate, $R(\mathbf{w})$ is the imposed L_1 restriction on the parameters, $R(\mathbf{w}) = \|\mathbf{w}\|_1 \in \mathbb{R}$, and λ is a positive real value. In fact, the sum term $\lambda R(\mathbf{w})$ is the negated log-likelihood obtained when we impose a centered Laplace distribution (with variance inversely proportionally to λ) on the components of $\mathbf{w}$. Consequently, this constraint imposes sparsity on the $\mathbf{w}$ components, and the parameter λ is controlling the regularization level on the estimate. Observe that parameter λ is a crucial selection to get a good solution. Notice that setting $\lambda = 0$ reverses the Lasso problem to minimizing the unregularized loss while a very large value will completely shrink the parameters to 0, leading to an empty model.

Based on the seminal paper of Tibshirani [26], different computational algorithms have been developed to solve the regularization path. Osborne et al. [29,30] developed the homotopy method for the squared loss function case and Efron et al. the LARS method [31].

Lasso has been successfully applied to multiple disciplines. In the bioinformatics field Vert et al. [32] adapted the Lasso sparse modeling to the problem of siRNA efficacy prediction. Tibshirani et al. [33] applied their fussed Lasso algorithm to protein mass spectroscopy and gene expression data, while Ghosh and Chinnaiyan [34] adapted the Lasso algorithm to the selection and classification of genomic biomarkers.

Finally, Zhao and Yu combine boosting and Lasso approximating the Lasso path in general situations. Their Boosted Lasso approach [27] has the computational advantages of Boosting converging to the Lasso solution with global L_1 regularization. This methodology has been recently applied to text classification [35] (reporting a significant improvement with respect to classic parameter estimation), handwritten character recognition [36], and intestinal motility analysis [37].

2.3 The Boosted Lasso Algorithm (BLasso)

The Boosted Lasso (BLasso) algorithm is a numerical method developed by Zhao and Yu [27] that finds a solution for general Lasso problems. Moreover, the algorithm can automatically tune the parameter λ. This method consists of a forward step similar to the statistical interpretation of a well-known learning method, Boosting [38], where an additive logistic regression model is sought using a maximum likelihood approach as a criterion. Moreover, there is a backward step that makes the algorithm able to correct mistakes made in early stages. This step uses the same minimization rule as the forward step to define each fitting stage with an additional restriction that forces the model complexity to decrease. The algorithm is specified in table 1. The constant ϵ, which represents the step size, controls the fineness of the grid BLasso runs on. On the other hand, the tolerance ξ controls how big a descend needs to be made for a backward step to be taken. This parameter should be much smaller than ϵ to have a good approximation and can be even set to 0 when the base learners set is finite. In that case, there is a theorem stated and proved in [27] that guarantees the convergence of the algorithm to the Lasso path when ϵ tends to 0 and the empirical loss $L(D, \mathbf{w})$ is strictly convex and continuously differentiable in $\mathbf{w}$.

BLasso can be easily extended to deal with convex penalties other than the L_1 restriction. This extended version, that is known as the Generalized Boosted Lasso algorithm, was also proposed by Zhao and Yu [27] and the procedure of the system remains the same. However, for general convex penalties, a step on different coordinates does not necessarily have the same impact on the penalty. For this reason one is forced to work directly with the penalized function, what generates small changes in the resulting algorithm as can be seen in [27].

2.4 The Laplacean Relevance Vector Machine (L-RVM)

As stated in section 2.1, a new formulation of the RVM is done by the optimization of the global loss function

$$G(\mathbf{w}) = L(D, \mathbf{w}) + \lambda R(\mathbf{w}) \tag{6}$$

being $L(D, \mathbf{w})$ the negated log-likelihood estimator for the parameters set $\mathbf{w}$ (see equation 3), and $R(\mathbf{w}) = \|\mathbf{w}\|_1$ a constraint to promote sparsity on $\mathbf{w}$. Given that this optimization corresponds to a Lasso problem, we

Table 1 Boosted Lasso Algorithm(BLasso)

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Notation

$v_a$, $a \in A = \{1, ..., d\}$ is a vector with having all 0s except a 1 in the position $a$.

Inputs:

  – $\varepsilon > 0$: small step size
  – $\xi \geq 0$: small tolerance parameter.

Initialization

  – $(\hat{a}, \hat{s}) \arg\min_{a \in A, s=\pm\varepsilon} \sum_{i=1}^{n} L(Z_i; s v_a)$
  – $\hat{\mathbf{w}}^0 = \hat{s} v_{\hat{a}}$
  – $\lambda^0 = \frac{1}{\varepsilon}(\sum_{i=1}^{n} L(Z_i; 0) - \sum_{i=1}^{n} L(Z_i; \hat{\mathbf{w}}^0))$
  – $I^0 = \{\hat{a}\}$
  – $r = 0$

Iterative Step

  – Compute $(\hat{a}) \arg\min_{a \in (I^r)} \sum_{i=1}^{n} L(Z_i; \hat{\mathbf{w}}^r + s_a v_a)$ where $s_a = -sign(\hat{\mathbf{w}}_a^r)$
  – If $G(\hat{B}^r + \hat{s}_{\hat{a}} 1_{\hat{j}}, \lambda^r) - G(\hat{\mathbf{w}}^r, \lambda^r) < -\xi$ then
      – $\hat{\mathbf{w}}^{r+1} = \hat{\mathbf{w}}^r + \hat{s}_{\hat{a}} 1_{\hat{j}}$ and $\lambda^{r+1} = \lambda^r$.
  – Otherwise
      – $(\hat{a}, \hat{s}) = \arg\min_{a \in A, s=\pm\varepsilon} \sum_{i=1}^{n} L(Z_i; \hat{\mathbf{w}}^r + s v_a)$
      – $\hat{\mathbf{w}}^{r+1} = \hat{\mathbf{w}}^r + \hat{s} v_{\hat{a}}$
      – $\lambda^{r+1} = \min\{\lambda^r, \frac{1}{\epsilon}(\sum_{i=1}^{n} L(Z_i; \hat{\mathbf{w}}^r) - \sum_{i=1}^{n} L(Z_i; \hat{\mathbf{w}}^{r+1}))$
      – $I^{r+1} = I^r \cup \{\hat{a}\}$

End: Stop when $\lambda^r \leq 0$.
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propose to minimize this function using the BLasso described in section 2.3. This new formulation of the RVM will be called the Laplacean Relevance Vector Machine and denoted by L-RVM.

3 Joint Feature Selection and Classifier Learning Method (JFSCL)

The classification accuracy of a classifier can be improved by performing an appropriate feature selection and determining which is the most suitable subset of variables to be considered during the classification process. For this purpose we extend the method proposed in section 2.4 by adding a new parameters vector that will control the activation of the relevant features. More concretely, we add a new sum term in the regularization part of the global loss function (6) obtaining a new cost function that can also be optimized using the described BLasso algorithm.

To control the activation of the features we consider a function $\sigma_k^d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined as follows

$$\sigma_k^d(\boldsymbol{a})(\sigma_k(a_1), ..., \sigma_k(a_d)), \forall \boldsymbol{a}(a_1, ..., a_d) \in \mathbb{R}^d \tag{7}$$

where k is any positive real value and $\sigma_k : \mathbb{R} \rightarrow \mathbb{R}$ is the sigmoid function,

$$\sigma_k(a) = \frac{1}{1 + \exp{(-ka)}}, \forall a \in \mathbb{R} \tag{8}$$

Let be $\mathbf{v} = (v_1, ..., v_d)$ a new parameters vector and $\mathbf{x} \in \mathbb{R}^d$ an input data. Then consider the following expression

$$\sigma_k^d(\mathbf{v}) \odot \mathbf{x} : (\sigma_k(v_1)x_1, ..., \sigma_k(v_d)x_d) \tag{9}$$

where $\odot$ denotes the Hadamard product. Observe, from the definition of the sigmoid function (equation 7), that the components of $\sigma_k^d(\mathbf{v})$ take, in general, values very close to 0 or very close to 1, supposing that k is high enough. Thus, equation (9) can express an activation or a deactivation of the i^{th} feature by its corresponding parameter v_i.

After these definitions and observations we can consider a new global loss function for the RVM that is constrained by an extended parameter set $\mathit{\Omega} = (\mathbf{w}, \mathbf{v})$:

$$\mathit{\Gamma}(\mathit{\Omega}) = L(D, \mathbf{w}, \mathbf{v}) + \lambda_1 \|\mathbf{w}\|_1 + \lambda_2 \|\sigma_k^d(\mathbf{v})\|_1 \tag{10}$$

This loss function represents a preference for solutions that use a small set of components from a small set of samples. In this case, a reasonable choice is to impose the same value for both regularization terms, $\lambda := \lambda_1 = \lambda_2$. Therefore we obtain the following expression of the loss:

$$\mathit{\Gamma}(\mathit{\Omega}) = L(D, \mathbf{w}, \mathbf{v}) + \lambda(\|\mathbf{w}\|_1 + \|\sigma_k^d(\mathbf{v})\|_1) \tag{11}$$

that can be minimized using the extended version of BLasso, the above mentioned Generalized Boosted Lasso. This approach will be called the Joint Feature Selection and Classifier Learning method (JFSCL).

4 Experiments

In this section we evaluate different aspects of the proposed methodologies. For this aim we use synthetic data and three public databases from the UCI Machine Learning Repository [39] that are binary: the PIMA Dabetes, the Breast Cancer and the Heart disease.

First of all we evaluate the performance of the proposed L-RVM and we compare it with the classical RVM approach. After that, we test the proposed JFSCL using synthetic data and perform some experiments to evaluate the sensitivity of the method to the design parameters. Finally we test the performance of the JFSCL with real data and compare this proposed method with other feature selection filters.

4.1 Classical RVM and L-RVM

In this section we will test the performance of the L-RVM that has been proposed in section 2.4. We have used a Gaussian kernel function $K(x_1, x_2)$:

$$K(x_1, x_2) = \exp(-\gamma \|x_1 - x_2\|^2) \tag{12}$$

being γ an appropriate positive real value that has been determined by cross validation.

We have performed experiments with three public databases from the UCI Machine Learning Repository, the PIMA Diabetes, the Breast Cancer, and the Heart disease data. To train and test the methods we have used public data sets from *http://ida.first.fraunhofer.de/projects/bench/benchmarks.htm*

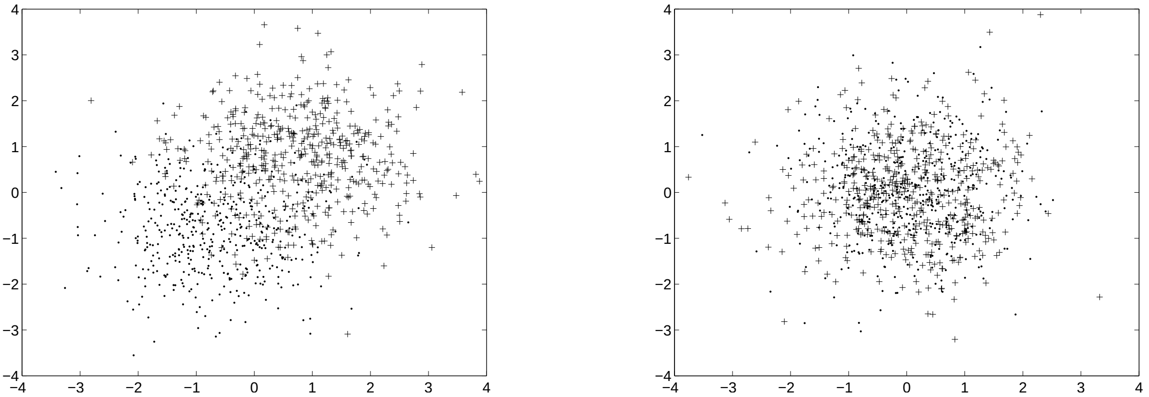
Table 2 Comparison of classification performance between RVM and L-RVM. For each used database we show the number of training samples N, the number of testing samples $Ntest$, the number of features d, the mean error rates, and the 95% confidence intervals for the RVM and L-RVM algorithms. The results are computed performing 100 runs with different splits of train and test sets.

Data Set	N	$Ntest$	d	**RVM**	**L-RVM**
PIMA Diabetes	468	300	8	24.10 ± 1.06	23.98± 0.58
Breast Cancer	200	77	9	26.04 ± 4.74	26.30 ± 0.80
Heart	170	100	13	16.34 ± 3.53	16.30 ± 0.68

and the obtained results are specified in table 2. These rates show that the proposed RVM and L-RVM are equivalent in terms of classification, being the average error rates overlapped in terms of statistical significance. Notice, however, that the confidence intervals of the L-RVM are smaller than the RVM ones, what suggest that the proposed L-RVM approach is more stable. The inputs of the method were $\epsilon = 1$ and $\xi = 10^{-5}$ and have been also determined by cross validation.

4.2 JFSCL for synthetic data

In this section we propose two synthetic experiments to test the feature selection performance of the JFSCL. This first synthetic experiment was designed as follows: d-dimensional data from two Gaussian distributions were generated with means:



(a) Scatter of the two first features from the Gaussian set (b) Scatter of two noise features from the Gaussian set

Fig. 1 Scatter of samples from the synthetic Gaussian data set. Symbol (+) represents points from class 1 and symbol (.) represents points from class 2.

$$\mu_1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0] \tag{13}$$

$$\mu_2 = [-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0] \tag{14}$$

and standard deviation 1 for all the components (notice that the distance between the d-dimensional Gaussian distribution centroids is 2). According to the optimal linear classifier, the theoretical Bayes error can be computed exactly for these two sets, and its value is 15.19. In the experiment, 200 samples (100 from each class) have been used for training the classical RVM, the proposed L-RVM, and the proposed JFSCL. The test set was composed by 1000 samples (500 from each class). An example of the scatter of the first two features and two noise features is shown in Figure 1.

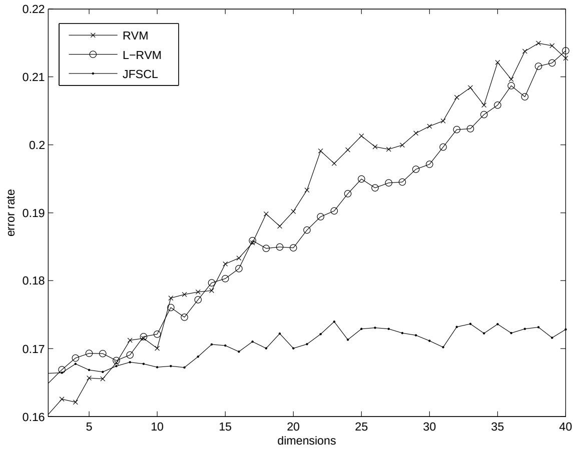


Fig. 2 Comparison of classification performance between RMV, L-RVM and JFSCL using the synthetic Gaussian data set. The X axis represents the data dimension, note that the only important dimensions are the two first ones, and the rest dimensions are Gaussian noise. The Y axis represent the obtained mean error rate over 20 runs.

Figure 2, shows the mean error rate of 20 random generations of the same experiment, as a function of the space dimensionality. As can be observed, the influence of adding spurious features on the RVM and L-RVM confuses the classifier, while our approach keeps a quite constant error rate that is close to the Bayes error. Moreover, our method focuses the classifier on the first two features (the ones with class separability information).

In addition, a second set of synthetic databases has been generated in order to test our proposal when data clusters are highly overlapped. Two d-dimensional Gaussian data clusters (one per class) were generated for $d = 2, 10, 20, 30, 40$ with means

Table 3 Mean error rate using the synthetic Gaussian data sets, varying the distance between the Gaussian centroids to get different overlapping levels between the classes. The error rate is shown as a function of both data dimensionality and inter-class centroid distance.

	L-RVM					JFSCL				
	Dimensionality					Dimensionality				
Centroid distance	**2**	**10**	**20**	**30**	**40**	**2**	**10**	**20**	**30**	**40**
0.25	41.09	43.09	44.92	45.52	46.25	42.68	42.61	43.82	44.33	44.92
0.5	32.24	33.87	35.46	36.35	38.26	32.05	33.42	33.48	34.24	34.29
0.75	23.42	24.27	26.13	27.06	28.74	23.19	23.44	23.87	23.93	24.47
1	15.94	17.49	19.21	20.07	21.20	15.78	16.16	16.71	16.73	17.22

$$\mu_1 = [\mu_{11}, \mu_{12}, 0, \ldots, 0] \tag{15}$$

$$\mu_2 = [\mu_{21}, \mu_{22}, 0, \ldots, 0] \tag{16}$$

and standard deviation 1 for all the components. We vary the distance between the centroids (μ_{11}, μ_{12}) and (μ_{21}, μ_{22}) from 0.25 to 1, having thus different class overlapping levels. The experiments have been performed with the L-RVM classifier and our proposed JFSCL method, and results can be seen in table 3. Notice that our proposal fares significantly better in almost all the cases, specially when the data dimensionality is high.

4.3 Analyzing the JFSCL Sensitivity to the Parameters Selection

In the proposed JFSCL there are two parameters that have to be fixed: (i) k, that controls the steepness of the sigmoid function, and (ii) ϵ, the step of the BLasso algorithm. Another parameter of the BLasso is the tolerance ξ, but in all the cases it has been fixed at 10^{-5} and small variations of this parameter do not affect the JSFCL.

In order to analyse the sensitiveness of the classification performance to variations of these design parameters, we have performed the same experiment for different values of k and ϵ. In particular we have performed these experiments with the 40-dimensional train and test sets used in section 4.2, having in this case 38 spurious components. These data were composed by 20 pairs of train and test sets, having 200 samples (100 per class) and 1000 samples (500 per class) respectively.

The means of the obtained error rates are shown in table 4. Notice that there is no much variation in the results, although in this case $\epsilon = 1.2$ and $k = 10$ is the best option. Moreover, in all the cases except for $\{k = 1, \epsilon = 0.4\}$ the obtained results are better than in the case of RVM (see this results in figure 2). In these experiments the worst parameters combination in terms of accuracy is $\{k = 1, \epsilon = 0.4\}$. We think that here the main problem is that the method can not significantly activate and/or deactivate a feature in one step. That suggest that there have to be a compromise between these two parameters. On the other hand, for ϵ greater than 50 the method do not converge for any of the considered k values.

Table 4 Mean error rates obtained in the classification of the 40-dimensional synthetic data with different values for the design parameters of the method. Parameter ϵ corresponds to the step parameter in the BLasso algorithm, and the k is the parameter of Eq.7

	$k = 1$	$k = 10$	$k = 20$	$k = 40$	$k = 80$	$k = 100$	$k = 500$
$\epsilon = 0.4$	26.39	17.59	17.66	17.20	16.58	17.44	17.61
$\epsilon = 0.8$	17.20	17.36	17.98	16.31	16.53	16.89	17.13
$\epsilon = 1.2$	16.77	16.01	16.41	16.48	16.40	17.02	18.56
$\epsilon = 2$	17.25	16.92	17.15	17.18	16.99	17.35	16.98
$\epsilon = 5$	17.42	17.52	16.40	16.62	16.70	16.39	17.38
$\epsilon = 10$	17.97	16.91	17.08	17.09	17.71	19.39	17.91

4.4 UCI Machine Learning Repository using the proposed JFSCL

In this section we test the proposed JFSCL method using the PIMA Diabetes, the Breast Cancer and the Heart disease databases. The train and test data sets are exactly the same as in section 4.1 to be able to compare the error rate obtained when classical RVM or L-RVM are used. The results are shown in table 5, where the mean in percentage of rejected features is also specified. We can see that the error rates are equivalent to the cases considered in section 4.1, comparing both tables 5 and 2. Nevertheless, the JFSCL method allows us to reduce the number of used features to the half of the original set.

Table 5 Obtained results (mean error rate and 95% confidence interval) using the JFSCL, and mean number of rejected features (in percentage).

DataSet	JFSCL Error rate	Rejected Features
PIMA Diabetes	24.23 ± 0.34	48.33%
Breast Cancer	26.52 ± 0.92	41.25%
Heart	16.28 ± 0.65	31.15%

4.5 Comparison with Feature Selection Filters

In order to test the influence of the embedded feature selection in our JFSCL method, we performed the same experiments using other state-of-the-art feature selection techniques. In all the cases we ran the feature selection algorithm selecting from one up to the maximum number of features (all the possibilities) and then performed the classification with the L-RVM classifier.

We have applied the following feature selection filters: (i) Forward Feature Selection (FFS) [40],(ii) Backward Feature Selection (BFS) [6], (iii) Feature Ranking (FR) using 1-Nearest Neighbour leave-one-out [12], and Floating Search (FS) [41].

The results obtained in these tests are shown in tables 6, 7, 8. Notice that in the experiments performed with the Heart database, the JFSCL performs always better than the other methods. However, with the other databases there are methods that perform as well as the proposed JFSCL. For instance, with the PIMA Diabetes database the best accuracy is obtained with Feature Ranking using 5 features, being the error 23.74%. Neverthe-

Table 6 Obtained results (mean error and 95% confidence interval) using the state-of-the-art feature selection methods with PIMA Diabetes Database. Notice that the proposed JFSCL method selected in this case 51.66% of the features (between 4 and 5 components) and obtained an error of 24.23 ± 0.34.

	1	2	3	4	5	6	7
FSF	33.11	25.19	24.46	24.48	24.51	24.15	24.40
	± 0.57	± 0.49	± 0.43	± 0.37	± 0.38	± 0.35	± 0.39
BSF	30.66	25.38	24.98	24.66	24.56	24.31	24.26
	± 0.78	± 0.45	± 0.51	± 0.37	± 0.43	± 0.36	± 0.36
FR	33.11	28.59	23.90	23.37	23.74	23.95	24.20
	± 0.57	± 0.71	± 0.52	± 0.32	± 0.32	± 0.33	± 0.32
FS	26.46	26.43	25.86	24.91	24.60	24.78	24.66
	± 0.24	± 0.27	± 0.28	± 0.23	± 0.24	± 0.22	± 0.21

less, although this result is slightly better that the 24.23% obtained with the JFSCL, the difference is not statistically significant, given that the confidence intervals are overlapped. On the other hand, notice that the tested state-of-the-art feature selection methods are not able to automatically fix the number of needed features, while the proposed JFSCL can automatically select both the features and the number of features. Moreover, these results suggest that the number of features selected by the JFSCL is the most appropriated. For instance, in the case of PIMA Diabetes Database the best accuracy is obtained using 5 features and the JFSCL selects in this case between 4 and 5 features. A similar behavior can be observed in the results obtained with the other databases.

Table 7 Obtained results (mean error and 95% confidence interval) using the state-of-the-art feature selection methods with Breast Cancer Database. Notice that the proposed JFSCL method selected in this case 58.75% of the features (between 5 and 6 components) and obtained an error of 26.52 ± 0.92.

	1	2	3	4	5	6	7	8
FSF	28.80	28.90	29.16	29.31	27.17	26.40	26.13	25.86
	± 0.90	± 0.90	± 0.87	± 0.91	± 1.00	± 0.95	± 0.83	± 0.89
BSF	28.82	29.07	29.56	28.38	27.31	26.78	26.23	25.54
	± 0.89	± 0.89	± 0.89	± 0.97	± 0.94	± 0.90	± 0.86	± 0.94
FR	28.80	28.80	28.80	28.32	26.58	25.21	25.21	25.61
	± 0.90	± 0.90	± 0.90	± 0.90	± 0.92	± 0.91	± 0.89	± 0.84
FS	28.81	25.16	24.83	25.48	26.04	26.30	26.62	26.62
	± 0.90	± 0.96	± 0.82	± 0.82	± 0.82	± 0.97	± 0.82	± 0.87

5 Conclusions

In this paper we present a new approach for the RVM classifier, the L-RVM, that is based on the Lasso problem. In concrete, we learn this L-RVM minimizing an empirical loss with L_1 penalty using the BLasso algorithm. Moreover, we extend this methodology to an embedded system for Joint Feature Selection and Classifier Learning (JFSCL).

We have performed several experiments to test both L-RVM and JFSCL.

Table 8 Obtained results (mean error and 95% confidence interval) using the state-of-the-art feature selection methods with Heart Database. Notice that the proposed JFSCL method selected in this case 68.85% of the features (between 8 and 9 components) and obtained an error of 16.28 ± 0.65.

	1	2	3	4	5	6	7	8	9	10	11	12
FSF	42.84	30.39	26.44	24.08	22.22	20.64	19.83	19.78	19.11	18.72	18.52	17.90
	± 1.11	± 1.10	± 0.94	± 0.76	± 0.85	± 0.73	± 0.73	± 0.66	± 0.71	± 0.69	± 0.74	± 0.74
BSF	38.90	29.35	24.35	21.07	20.10	19.80	19.25	18.76	18.69	18.08	18.19	17.76
	± 1.65	± 1.08	± 0.80	± 0.86	± 0.79	± 0.75	± 0.70	± 0.73	± 0.75	± 0.73	± 0.75	± 0.68
FR	42.84	37.30	33.35	31.08	28.16	24.40	23.31	24.31	23.54	21.97	19.27	18.79
	± 1.11	± 1.22	± 0.97	± 0.96	± 0.92	± 0.83	± 0.85	± 0.78	± 0.81	± 1.00	± 0.88	± 0.80
FS	23.47	24.32	19.75	18.08	17.65	17.52	16.90	16.94	16.98	16.76	16.96	17.11
	± 0.70	± 0.79	± 0.79	± 0.78	± 0.75	± 0.79	± 0.69	± 0.69	± 0.75	± 0.75	± 0.67	± 0.68

5.1 Discussion

In our experiments we have seen that the proposed L-RVM can performs as well as the classical RVM approach. The main advantages of the L-RVM is that we do not have to compute any derivative of the model and, moreover, the approach can be easily extended imposing other constraints on the parameters set, for instance.

On the other hand, we have tested the JFSCL using both synthetic data and real data. In the case of the synthetic data, where the problem is specially designed according the assumptions of the method, we can see that the JFSCL performs considerably better than the L-RVM or the classical

RVM. However, this difference is not evident in the case of real data. In this second case the JFSCL can perform as well as the L-RVM while it uses less features (just the selected ones).

Furthermore, we have compared the JFSCL with other state-of-the-art feature selection filters. Although some of these methods can perform quite well with the tested databases, the main drawback of them is that they take as input the number of features to select. However, the proposed JFSCL does it automatically. We have performed the experiments with the feature selection filters, considering all the possibilities: selection from 1 up to $d-1$ features, being d the data dimensionality. Two important observations have to be considered from these experiments:

- the quantity of features selected by the JFSCL coincides, usually, with the quantity of features that achieve the best accuracy with the other methods
- in terms of accuracy, the JFSCL is in some cases the best result and, in other cases, is very close to the best rate. However, we want to emphasize that the other filters need to try all the possible feature quantity, having thus some advantage in comparison with the JFSCL.

Finally, another important issue of the proposed JFSCL is the following: it can be seen as a meta-method to jointly learn a classifier and, at the same time, select the most appropriate features to consider. This methodology could be applied to other loss functions and classifiers. On the other hand, given the classifier embedded nature of the JFSCL, density estimations on the data sets are avoided.

5.2 Future work

The main drawback of the proposed JFSCL is its computational complexity. Notice that the likelihood function has to be evaluated $2m$ times at each iteration of the BLasso algorithm, being m the number of the model's parameters. However we plan to reduce this complexity using some techniques that can be found in the literature. For instance, a stochastic version could be developed or blockwise strategies might be applied.

On the other hand, the JFSCL can be extended to simultaneously perform a feature extraction task. Instead of using a vector of sigmoid selectors, we could use a complete discriminant projection matrix. In this context there would be 2 options: to fix in advance the projection matrix, or to properly combine the original features promoting sparsity with a Laplace prior.

Moreover, the sparsity on both the model and feature selection tasks are imposed using as regularization the L_1 of the parameters vector. Other norms could be used to enforce this sparsity, such as $L_{\frac{1}{2}}$.

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