

Representing Upper Probability Measures over Rational Łukasiewicz Logic

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Abstract

Upper probability measures are measures of uncertainty that generalize probability measures in order to deal with non-measurable events. Following an approach that goes back to previous works by Hájek, Esteva, and Godo, we show how to expand Rational Łukasiewicz Logic by modal operators v in order to reason about upper probabilities of classical Boolean events φ so that $v(\varphi)$ can be read as “the upper probability of φ ”. We build the logic $\mathcal{U}(\text{RL})$ for representing upper probabilities and show it to be complete w.r.t. a class of Kripke structures equipped with an upper probability measure. Finally, we prove that the set of $\mathcal{U}(\text{RL})$ -satisfiable formulas is NP-complete.

1 Introduction

Probability measures are a common tool in the formalization and quantification of our degrees of confidence in the occurrence of some events. However, classical probability theory seems to fail in formalizing certain situations when the information is not complete.

Consider the following case. Suppose we have a box with 100 balls inside. We know that there are 50 red balls, while the remaining balls are either blue or green. What is the probability to pick up a green ball? We know exactly how many the red balls are, so we know that the probability of picking up a red ball is 0.5, and so is the probability of picking up either a blue or a green ball. But we do not have sufficient information about the distribution of green and blue balls to be able to know the exact probability of picking up a ball of one of those colors. The events “picking up a blue ball” and “picking up a green ball” are non-measurable from the point of view of probability theory. However, we have enough information to know that the probability of those events may vary in an interval bounded by a maximum and a minimum probability. We know for sure that the probability

α of picking up a blue ball is somewhere between $[0, 0.5]$, and, consequently, the probability of picking up a green ball lies in the interval $[0, 0.5 - \alpha]$. Then, we can represent the uncertainty of the situation at hand by means of a set of probability measures $\mathcal{P} = \{\mu_\alpha : \alpha \in [0, 0.5]\}$, such that the probability of “picking up a red ball” is 0.5, the probability of “picking up a blue ball” is α , and the probability of “picking up a green ball” is $0.5 - \alpha$. Given the set \mathcal{P} we can then determine the best and worst estimate for an event φ by taking the supremum and the infimum of $\mu_\alpha(\varphi)$ in \mathcal{P} .

This can be formalized as follows (see [22]). Given a set of probability measures μ_i over the same Boolean algebra, the *upper probability* $\pi(\varphi)$ is defined as $\sup\{\mu_i(\varphi)\}$ and the *lower probability* $\lambda(\varphi)$ is defined as $\inf\{\mu_i(\varphi)\}$. Upper and lower probabilities are dual, since from an upper probability we can define a lower probability as follows: $\lambda(\varphi) = 1 - \pi(\neg\varphi)$, and viceversa.

Upper and lower probabilities can be also seen as classes of fuzzy measures. Recall that, given a set W of possible situations, a *fuzzy measure* [21] is a mapping ϕ from the Boolean algebra of subsets of W into the real unit interval $[0, 1]$ satisfying the following properties: (i) $\phi(\perp) = 0$, (ii) $\phi(\top) = 1$, (iii) if $\vdash \varphi \rightarrow \psi$ then $\phi(\varphi) \leq \phi(\psi)$. As shown by Anger and Lembcke in [1], any upper probability is a fuzzy measure π such that for all natural numbers m, n, k , and all $\varphi_1, \dots, \varphi_m$, if $\{\{\varphi_1, \dots, \varphi_m\}\}$ is an (n, k) -cover¹ of (φ, \top) , then

$$(\#) \quad k + n\pi(\varphi) \leq \sum_{i=1}^m \pi(\varphi_i).$$

Halpern and Pucella proved in [14] that when the sample space is finite there are only finitely many instances of the above property. Indeed, there exist constants k_0, k_1, \dots such that if W is a finite set, for all natural numbers $m, n, k \leq k_{|W|}$, and all $\varphi_1, \dots, \varphi_m$, if $\{\{\varphi_1, \dots, \varphi_m\}\}$ is an (n, k) -cover of (φ, \top) , then $(\#)$ holds.

Similarly we can see any lower probability as a fuzzy measure λ such that for all natural numbers m, n, k , and all $\varphi_1, \dots, \varphi_m$, if $\{\{\varphi_1, \dots, \varphi_m\}\}$ is an (n, k) -cover of (φ, \top) , then

$$(\#\#) \quad k + n\lambda(\varphi) \geq \sum_{i=1}^m \lambda(\varphi_i).$$

Halpern and Pucella studied in [14] a logic for reasoning about upper probabilities extending classical logic by a probabilistic modal operator whose interpretation corresponds to an upper probability measure. In this work, we aim at providing an alternative treatment for the logical representation of upper probabilities by taking a different approach.

Esteva, Hájek, and Godo proposed in [12, 8] a new interpretation of measures of uncertainty in the framework of t-norm based logics [11, 10]. Given a sentence as “The proposition φ is probable (believable, plausible)”, its degrees of truth

¹A proposition φ is said to be *covered* n times by a multiset $\{\{\varphi_1, \dots, \varphi_m\}\}$ of propositions, if every situation in which φ is true makes true at least n propositions from $\varphi_1, \dots, \varphi_m$ as well. An (n, k) -cover of (φ, \top) is a multiset $\{\{\varphi_1, \dots, \varphi_m\}\}$ that covers \top k times and covers φ $n + k$ times.

can be interpreted as the degree of uncertainty of the proposition φ . Indeed, the higher is our degree of confidence in φ , the higher the degree of truth of the above sentence will be. In some sense, the predicate “is probable (believable, plausible)” can be regarded as a modal operator over the proposition φ . Then, given a measure of uncertainty ϕ , we can define modal many-valued formulas $\kappa(\varphi)$, whose interpretation is given by a real number corresponding to the degree of uncertainty assigned to φ under ϕ . Furthermore, we can translate the peculiar axioms governing the behavior of an uncertainty measure into formulas of a certain t-norm based logic, depending on the operations we need to represent.

In this work, we show how the above approach can be adapted to represent and reasoning about upper probability measures by relying on Rational Łukasiewicz Logic [6].

The paper is organized as follows. In the next section we briefly recall the main properties of Łukasiewicz logic and Rational Łukasiewicz logic. In Section 3, we introduce the logic $\mathcal{U}(\text{RL})$ for reasoning about upper probabilities and prove a completeness result. Finally, in Section 4, we study the computational complexity of the set of $\mathcal{U}(\text{RL})$ -satisfiable formulas and show that it is NP-complete. We end with some final remarks.

2 Rational Łukasiewicz Logic

Recall that Łukasiewicz logic \mathbf{L} (see [3]) is built up from the primitive connective \rightarrow and the truth constant $\bar{0}$. Further connectives are defined as follows:

$$\begin{array}{ll} \neg\varphi & \text{is } \varphi \rightarrow \bar{0}, & \varphi \wedge \psi & \text{is } \varphi \&(\varphi \rightarrow \psi), \\ \varphi \&\psi & \text{is } \neg(\varphi \rightarrow \neg\psi), & \varphi \oplus \psi & \text{is } \neg(\neg\varphi \&\neg\psi), \\ \varphi \vee \psi & \text{is } ((\varphi \rightarrow \psi) \rightarrow \psi), & \varphi \leftrightarrow \psi & \text{is } (\varphi \rightarrow \psi) \&(\psi \rightarrow \varphi). \end{array}$$

The axioms of Łukasiewicz logic are the following:

$$\begin{array}{ll} (\mathbf{L1}) & \varphi \rightarrow (\psi \rightarrow \varphi), & (\mathbf{L2}) & (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)), \\ (\mathbf{L3}) & (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi), & (\mathbf{L4}) & ((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi). \end{array}$$

The only inference rule is *modus ponens*, i.e.: from $\varphi \rightarrow \psi$ and φ derive ψ .

A *proof* in \mathbf{L} is a sequence $\varphi_1, \dots, \varphi_n$ of formulas such that each φ_i either is an axiom of \mathbf{L} or follows from some preceding φ_j, φ_k ($j, k < i$) by modus ponens. As usual, a set of formulas is called a theory. We say that a formula φ can be derived from a theory T , denoted as $T \vdash \varphi$, if there is a proof of φ from a set $T' \subseteq T$. A theory T is said to be consistent if $T \not\vdash \bar{0}$.

The algebraic semantics for Łukasiewicz logic is given by MV-algebras [3], i.e. structures $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ satisfying the following equations:

$$\begin{array}{ll} (\text{MV1}) & x \oplus (y \oplus z) = (x \oplus y) \oplus z, & (\text{MV2}) & x \oplus y = y \oplus x, \\ (\text{MV3}) & x \oplus 0 = x, & (\text{MV4}) & \neg\neg x = x, \\ (\text{MV5}) & x \oplus \neg 0 = 0, & (\text{MV6}) & \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x. \end{array}$$

The class of MV-algebras forms a variety \mathbf{MV} that also is the equivalent algebraic semantics for Łukasiewicz logic, in the sense of Blok and Pigozzi [2]. As

shown by Chang, MV is generated as a quasivariety by the standard MV-algebra, i.e. the MV-algebra over the real unit interval $[0, 1]$, where $x \oplus y = \min(x + y, 1)$, and $\neg x = 1 - x$.

An evaluation e for Lukasiewicz logic's formulas into the standard MV-algebra is a mapping $e : Form \rightarrow [0, 1]$ assigning to all propositional variables a value from the real unit interval, that can be extended to compound formulas as follows:

$$\begin{aligned} e(\neg\varphi) &= 1 - e(\varphi), & e(\varphi \leftrightarrow \psi) &= 1 - |e(\varphi) - e(\psi)|, \\ e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)), & e(\varphi \vee \psi) &= \max(e(\varphi), e(\psi)), \\ e(\varphi \oplus \psi) &= \min(1, e(\varphi) + e(\psi)), & e(\varphi \&\psi) &= \max(0, e(\varphi) + e(\psi) - 1). \end{aligned}$$

An evaluation e is a *model* for a formula φ if $e(\varphi) = 1$. An evaluation e is a model for a theory T , if $e(\psi) = 1$, for every $\psi \in T$.

The fact that the MV is the equivalent algebraic semantics for Lukasiewicz logic and is generated as a quasivariety by the standard MV-algebra implies that Lukasiewicz logic is finitely strongly standard complete, i.e.: for every finite theory T and every formula φ , $T \vdash \varphi$ iff every model e of T also is a model of φ .

A very well-known result by McNaughton [19] states that the free MV-algebra over n -generators coincides with the set of continuous piecewise linear polynomial functions with integer coefficients over the n th-cube. In other words, every continuous piecewise linear polynomial function with integer coefficients over $[0, 1]^n$ is definable by a term in Lukasiewicz logic.

Rational Lukasiewicz logic RL is an expansion Lukasiewicz logic introduced by Gerla in [6], obtained by adding the unary connectives δ_n , for each $n \in \mathbb{N}$, plus the following axioms:

$$(D1) \underbrace{\delta_n \varphi \oplus \dots \oplus \delta_n \varphi}_n \leftrightarrow \varphi, \quad (D2) \neg \delta_n \varphi \oplus \neg \underbrace{(\delta_n \varphi \oplus \dots \oplus \delta_n \varphi)}_{n-1}.$$

The algebraic semantics for RL is given by DMV-algebras (divisible MV-algebras), i.e. structures $\mathcal{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n \in \mathbb{N}}, 0 \rangle$ such that $\langle A, \oplus, \neg, 0 \rangle$ is an MV-algebra and the following equations hold for all $x \in A$ and $n \in \mathbb{N}$:

$$(\delta_n 1) \quad n.\delta_n x = x, \quad (\delta_n 2) \quad \delta_n x * (n-1).\delta_n x = 0,$$

where by $n.x$ we denote the element of A inductively defined by $0.x = 0$, $(n-1).x = n.x \oplus x$. An evaluation into the real unit interval is extended for the connectives δ_n as follows: $e(\delta_n \varphi) = \frac{e(\varphi)}{n}$.

Notice that in RL we can define all the rationals in $[0, 1]$ in the following way:

- $\frac{1}{n}$ is given by $\delta_n \bar{1}$, i.e. $e(\delta_n \bar{1}) = \frac{1}{n} \cdot 1$,
- $\frac{m}{n}$ is given by $m.\delta_n \bar{1}$, i.e. $e(m.\delta_n \bar{1}) = \underbrace{\frac{1}{n} \oplus \dots \oplus \frac{1}{n}}_m$.

As shown in [6], the variety of DMV-algebras is generated as a quasivariety by the standard DMV-algebra over $[0, 1]$, and RL is finitely strongly standard complete. Moreover, the free DMV-algebra over n -generators coincides with the

set of continuous piecewise linear polynomial functions with rational coefficients over the n th-cube.

Finally, recall that the satisfiability problem for both Lukasiewicz logic and Rational Lukasiewicz logic is NP-complete, as proved [20] in and [6], respectively.

3 Reasoning about Upper Probabilities

In this section, we will build the logic $\mathcal{U}(\text{RL})$ for reasoning about upper probabilities by relying on RL . To begin, notice that the condition $(\#)$ is equivalent to

$$\frac{k}{m} + \frac{n}{m} \pi(\varphi) \leq \sum_{i=1}^m \frac{\pi(\varphi_i)}{m},$$

given that $n, k \leq m$. It is then clear that $\sum_{i=1}^m \frac{\pi(\varphi_i)}{m} \leq 1$, and so it makes sense to

rely on RL . Indeed, what we need in order to represent upper probability measures is to rely on a logic that allows the representation of rational numbers, the product of rationals and formulas, and the sum. Thus RL perfectly fits this description. Furthermore, the presence of the standard involutive negation makes possible to define also lower probabilities.

Notice that RL is not the only logic adequate for the representation of upper probabilities. For example, logics like RPPL'_{Δ} [15] and $\text{L}\Pi_{\frac{1}{2}}$ [4] both represent suitable choices. However, as far as we know, satisfiability for both RPPL'_{Δ} and $\text{L}\Pi_{\frac{1}{2}}$ is in PSPACE [13, 18], thus this does not currently allow us to obtain an NP-containment result for the logic for upper probabilities as shown in the next section.

$\mathcal{U}(\text{RL})$ is built over RL extending its language by including modal formulas whose interpretation corresponds to an upper probability measure. We define the language in two steps. First, we have classical Boolean formulas φ, ψ , etc., defined in the usual way from the classical connectives (\wedge, \neg) and from a countable set V of propositional variables p, q, \dots , etc. The set of Boolean formulas is denoted by L . Elementary modal sentences are formulas of the form $v(\varphi)$, where v is a unary operator taking as arguments Boolean sentences. Compound modal formulas are built by means of the RL -connectives. Nested modalities are not allowed.

The axioms of the logic $\mathcal{U}(\text{RL})$ are the following:

- (i) The set of classical Boolean tautologies
- (ii) Axioms of RL for modal formulas
- (iii) The following axiom:

$$(\pi 1) \quad \neg v(\perp).$$

Deduction rules of $\mathcal{U}(\text{RL})$ are those of RL , plus:

- (iv) *modalization*: from $\vdash \varphi$ (i.e. φ is derivable in Classical Logic) derive $v(\varphi)$
- (v) *monotonicity*: from $\vdash \varphi \rightarrow \psi$ derive $v(\varphi) \rightarrow v(\psi)$.

(v) *upper probability rule (UP)*: if $\varphi \rightarrow \bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k+n}} \bigwedge_{j \in J} \varphi_j$, and $\bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k}} \bigwedge_{j \in J} \varphi_j$ are propositional tautologies, then derive

$$k.\delta_m \oplus n.\delta_m v(\varphi) \rightarrow \bigoplus_{j=1}^m \delta_m v(\varphi_j).$$

We now define the semantics for $\mathcal{U}(\text{RL})$ by introducing Upper Probability Kripke structures.

An Upper Probability Kripke model is a structure $K = \langle W, \mathcal{U}, e, \pi \rangle$, where:

- W is a non-empty set of possible worlds.
- \mathcal{U} is a Boolean algebra of subsets of W .
- $e : V \times W \rightarrow \{0, 1\}$ is a *Boolean* evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. Any given truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ , we will denote by $[\varphi]_W$ the set of worlds in which φ is true, i.e. $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\pi : \mathcal{U} \rightarrow [0, 1]$ is an upper probability measure over \mathcal{U} , such that $[\varphi]_W$ is π -measurable for any non-modal φ .
- $e(\cdot, w)$ is extended to elementary modal formulas by defining $e(v(\varphi), w) = \pi([\varphi]_W)$, and to arbitrary modal formulas according to the RL -semantics.

A structure K is a model for Φ , written $K \models \Phi$, if $e^K(\Phi) = 1$. If T is a set of formulas, we say that K is a model of T if $K \models \Phi$ for all $\Phi \in T$. The notion of logical entailment relative to a class of structures \mathcal{K} , written $\models_{\mathcal{K}}$, is then defined as follows:

$$\Gamma \models_{\mathcal{K}} \Phi \text{ iff } K \models \Phi \text{ for each } K \in \mathcal{K} \text{ model of } \Gamma.$$

If \mathcal{K} denotes the whole class of Upper Probability Kripke structures we shall write $\Gamma \models_{\mathcal{U}(\text{RL})} \Phi$. When $\models_{\mathcal{K}} \Phi$ holds we will say that Φ is *valid* in \mathcal{K} , i.e. when Φ gets value 1 in all structures $K \in \mathcal{K}$.

Proposition 3.1 (Soundness) *The logic $\mathcal{U}(\text{RL})$ is sound with respect to the class of Upper Probability Kripke structures.*

Proof. We just have to show that $(\pi 1)$ is valid in the class of Upper Probability Kripke structures and that the rules of inference preserve validity in a model.

- $e(\pi 1) = 1$, given that $\pi(\perp) = 0$.
- As for the modalization rule, suppose that $K \models \varphi$, then $[\varphi]_W = W$. Hence $e(v(\varphi), w) = 1$, that is $K \models v(\varphi)$.
- As for the monotonicity rule, suppose that $K \models \varphi \rightarrow \psi$. Clearly we have that $e(v(\varphi)) = \pi(\varphi) \leq \pi(\psi) = e(v(\psi))$, hence $K \models v(\varphi) \rightarrow v(\psi)$.

- As for the upper probability rule, suppose that $\varphi \rightarrow \bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k+n}} \bigwedge_{j \in J} \varphi_j$, and $\bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k}} \bigwedge_{j \in J} \varphi_j$ are propositional tautologies. Then, for all natural numbers m, n, k , and all $\varphi_1, \dots, \varphi_m$, $\{\{\varphi_1, \dots, \varphi_m\}\}$ is an (n, k) -cover of (φ, \top) , and so

$$k + n\pi(\varphi) \leq \sum_{i=1}^m \pi(\varphi_i),$$

meaning that $e(k.\delta_m \oplus n.\delta_m v(\varphi) \rightarrow \bigoplus_{j=1}^m \delta_m v(\varphi_j)) = 1$.

■

For any $\varphi, \psi \in L$, define $\varphi \sim \psi$ iff $\varphi \leftrightarrow \psi$ is provable in classical propositional logic. The relation \sim is an equivalence relation in classical logic and $[\varphi]$ will denote the equivalence class of φ . Obviously, the quotient set L/\sim forms a Boolean algebra which is isomorphic to a subalgebra $\mathcal{B}(\Omega)$ of the power set of the set Ω of Boolean interpretations of the crisp language L . For each $\varphi \in L$, we shall identify the equivalence class $[\varphi]$ with the set $\{\omega \in \Omega \mid \omega(\varphi) = 1\} \in \mathcal{B}(\Omega)$ that makes φ true. We shall denote by \mathcal{M} the set of upper probability measures defined over L/\sim or, equivalently, on $\mathcal{B}(\Omega)$.

Notice that each upper probability measure $\pi \in \mathcal{M}$ induces an Upper Probability Kripke structure $\langle \Omega, \mathcal{B}(\Omega), e_\pi, \pi \rangle$ where $e_\pi(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega$ and each propositional variable p . Denote by \mathcal{K}^π the class of Upper Probability Kripke structures induced by $\pi \in \mathcal{M}$. Abusing the language, we will say that an upper probability measure $\pi \in \mathcal{M}$ is a *model* of a modal theory T whenever the induced Kripke structure $\langle \Omega, \mathcal{B}(\Omega), e_\pi, \pi \rangle$ is a model of T .

Given the above notions, we can now prove completeness for $\mathcal{U}(\text{RL})$.

Theorem 3.2 (Finite Strong completeness) *The logic $\mathcal{U}(\text{RL})$ is finitely strongly complete w.r.t. the class of Upper Probability Kripke models, i.e.*

$$T \vdash_{\mathcal{U}(\text{RL})} \Phi \text{ iff } e_\pi(\Phi) = 1$$

for each upper probability measure $\pi \in \mathcal{M}$ model of T .

Proof. The proof is a straightforward adaptation of the one given by Hájek in [11] for reasoning about probability, and can be seen as a special case of a more general proof for logics for fuzzy measures given in [17]. Then, we simply briefly sketch the main steps of the proof.

First, we translate theories over $\mathcal{U}(\text{RL})$ into theories over RL . We define a theory, called \mathcal{F} , as follows:

- (i) take as propositional variables of the theory variables of the form f_φ , where φ is a classical proposition from L ,
- (ii) take as axioms of the theory the following ones, for each φ and ψ :

- ($\mathcal{F}1$) f_φ , if φ is a classical tautology,
 ($\mathcal{F}2$) $f_\varphi \rightarrow f_\psi$, whenever $\varphi \rightarrow \psi$ is a classical tautology,
 ($\mathcal{F}3$) $\neg f_\perp$.
 ($\mathcal{F}4$) $k.\delta_m \oplus n.\delta_m f_\varphi \rightarrow \bigoplus_{j=1}^m \delta_m f_{\varphi_j}$, with

$$\varphi \rightarrow \bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k+n}} \bigwedge_{j \in J} \varphi_j, \quad \text{and} \quad \bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k}} \bigwedge_{j \in J} \varphi_j$$

being classical tautologies.

Then define the mapping \star from modal formulas to RL-formulas as follows:

- $(v(\varphi))^\star = f_\varphi$,
- $(\Phi \oplus \Psi)^\star = \Phi^\star \oplus \Psi^\star$,
- $(\natural(\Phi))^\star = \natural(\Phi^\star)$, for \natural being \neg or δ_n .

Let us denote by T^\star the set of all formulas translated from T .

Then, following [11], one can easily check that for any Φ ,

$$T \vdash_{\mathcal{U}(\text{RL})} \Phi \text{ iff } T^\star \cup \mathcal{F} \vdash_{\text{RL}} \Phi^\star. \quad (1)$$

Now we prove that the semantical analogue of (1) also holds, that is,

$$T \models_{\mathcal{U}(\text{RL})} \Phi \text{ iff } T^\star \cup \mathcal{F} \models_{\text{RL}} \Phi^\star. \quad (2)$$

Assume $T^\star \cup \mathcal{F} \not\models_{\text{RL}} \Phi^\star$. This means that there exists an RL-evaluation e which is model of $T^\star \cup \mathcal{F}$ such that $e(\Phi^\star) < 1$. Define an upper probability measure π_e on $\mathcal{B}(\Omega)$ as follows:

$$\pi_e([\varphi]) = e(f_\varphi).$$

Moreover, let

$$e'(p, w) = w(p)$$

for each propositional variable p . We prove that π_e is an upper probability measure by showing that, given e , the axioms of upper probabilities do hold.

- (i) By $\mathcal{F}1$ we have that for any Boolean tautology \top , $e(f_\top) = 1$. Then $\pi_e(\top) = 1$.
- (ii) By $\mathcal{F}2$ if $\varphi \rightarrow \chi$ is a classical tautology, then $f_\varphi \rightarrow f_\chi$ is an axiom. Consequently $\pi_e(\varphi) \leq \pi_e(\chi)$.
- (iii) Given $\mathcal{F}3$, $e(\neg f_\perp) = 1$. But, $e(\neg f_\perp) = e(f_\perp) \rightarrow 0$. Therefore, $e(f_\perp) \rightarrow 0 = 1$, which means $e(f_\perp) = 0$, and consequently $\pi_e(\perp) = 0$.
- (iv) By $\mathcal{F}4$ if $\varphi \rightarrow \bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k+n}} \bigwedge_{j \in J} \varphi_j$, and $\bigvee_{\substack{J \subseteq \{1, \dots, m\} \\ |J|=k}} \bigwedge_{j \in J} \varphi_j$ are classical tautologies, then $k.\delta_m \oplus n.\delta_m f_\varphi \rightarrow \bigoplus_{j=1}^m \delta_m f_{\varphi_j}$ is an axiom of \mathcal{F} , and so $k.\delta_m \oplus n.\delta_m \pi_e(\varphi) \rightarrow \bigoplus_{j=1}^m \delta_m \pi_e(\varphi_j)$ holds.

Therefore, we have proved that π_e actually is an upper probability measure. Then, it is clear that the model $K_e = \langle \Omega, \mathcal{B}(\Omega), \pi_e, e' \rangle$ is a model of T . Indeed, for any $w \in \Omega$, $e'(\varphi, w) = 1$ for any φ in T , and the truth degree of modal formulas Ψ coincides with the truth evaluation $e(\Psi^*)$ since it only depends on the values of π_e and e over the elementary modal formulas $\pi(\varphi)$ and atoms f_φ respectively. Therefore $e'(\Psi, w) = e(\Psi^*)$ for every modal formula Ψ , and in particular $e'(\Phi, w) = e(\Phi^*) < 1$.

Conversely, suppose that there is an Upper Probability Kripke structure $K = (W, \mathcal{U}, e, \pi)$ that is a model of T , but $K \not\models \Phi$. Take an arbitrary $w \in W$, and define:

$$e_K(f_\varphi) = e(v(\varphi), w) = \pi([\varphi]_W).$$

Clearly, e_K is a model of axioms $\mathcal{F}1 - \mathcal{F}4$ since π is an upper probability measure. Therefore $e_K(\Psi^*) = 1$ for every $\Psi^* \in T^* \cup \mathcal{F}$ but $e_K(\Phi^*) < 1$, as desired.

From (1) and (2), to prove the theorem it remains to show that

$$T^* \cup \mathcal{F} \vdash_{\text{RL}} \Phi^* \text{ iff } T^* \cup \mathcal{F} \models_{\text{RL}} \Phi^*.$$

However, the above equivalence in general does not hold, since \mathcal{F} contains infinitely many instance of axioms $\mathcal{F}1 - \mathcal{F}4$, and RL is not standard complete w.r.t. infinite theories. The solution is to take disjunctive normal forms exactly as done in [11]. Indeed, we can replace the infinitely many formulas in \mathcal{F} by finitely many instances of those axioms by substituting to each atom f_φ its corresponding disjunctive normal form built from the propositional variables appearing in T .

Still, we have to be careful, since $\mathcal{F}4$ holds for all $n, m, k \in \mathbb{N}$. However, notice that there are finitely many propositional variables in T , and so the related Boolean algebra of provably equivalent propositions has finitely many atoms. Then, as proven by Halpern and Pucella (see above and [14]), there are only finitely many instances of (\sharp) , and similarly there are only finitely many instances of $\mathcal{F}4$, in which we can substitute disjunctive normal forms. Then, the whole theory can be reduced to a finite set. The rest of the proof proceeds exactly as in [11, 17]. ■

4 Computational Complexity

In this section we will show that the logic $\mathcal{U}(\text{RL})$ is decidable, and the problem of checking the satisfiability for its formulas is NP-complete. In order to do so, we will rely on the logic AX^{up} introduced by Halpern and Pucella, who also showed that satisfiability for AX^{up} is NP-complete. We will show that the problem of checking satisfiability for formulas of $\mathcal{U}(\text{RL})$ is reducible to checking the satisfiability of a formula in AX^{up} .

We begin by briefly describing the main properties of AX^{up} . The language of AX^{up} is built from a set of classical propositional variables p_1, p_2, \dots closed under \neg and \rightarrow . A modal operator ℓ , standing for “likelihood”, is applied over Boolean formulas, so that $\ell(\varphi)$ is a likelihood term interpreted as “the upper probability of φ ”. A basic likelihood formula is an expression of the form

$$a_1\ell(\varphi_1) + \cdots + a_k\ell(\varphi_k) > b,$$

where a_1, \dots, a_k, b are rational numbers and $k \geq 1$ ². Likelihood formulas are Boolean combinations of basic likelihood formulas.

The semantics for AX^{up} is given by upper probability structures that are Kripke models equipped with an upper probability measure π . A basic likelihood formula $a_1\ell(\varphi_1) + \cdots + a_k\ell(\varphi_k) > b$, is satisfiable in a model K iff $a_1\pi([\varphi_1]_K) + \cdots + a_k\pi([\varphi_k]_K) > b$, where $[\varphi_i]_K$ is the set of worlds in the model in which φ_i is true. Satisfiability of Boolean combinations of basic likelihood formulas is obviously defined.

Halpern and Pucella showed that AX^{up} is complete w.r.t. interpretations into the above class of upper probability structures. Moreover, they showed that the problem of checking satisfiability for AX^{up} is NP-complete. By relying on the above result, we will prove that also satisfiability in $\mathcal{U}(\text{RL})$ is NP-complete:

Theorem 4.1 *The set of $\mathcal{U}(\text{RL})$ -satisfiable formulas is NP-complete.*

Proof. For the sake of simplicity we assume that all the $\mathcal{U}(\text{RL})$ -probabilistic formulas are combinations of elementary probabilistic formulas in the language $\langle \oplus, \neg, \{\delta_n\}, 0 \rangle$.

To prove hardness just notice that a classical formula φ is satisfiable iff so is $v(\varphi)$ in $\mathcal{U}(\text{RL})$.

Now, we prove NP-containment. We begin by showing how to translate a probabilistic formula Φ into a Boolean combination of linear polynomial equalities and inequalities. Let $S = \{\Psi_1, \dots, \Psi_m\}$ be the set of all subterms of Φ , with Ψ_m corresponding to Φ . Clearly the length of S is linear in the number of subterms of Φ .

Now, to subterms Ψ_i, Ψ_j associate variables x_φ^i, y^j , so that if Ψ_i is an elementary probabilistic formula $\pi(\varphi)$, then $\Psi_i \mapsto x_\varphi^i$, while if Ψ_j is a complex modal formula $\Psi_j \mapsto y^j$. For each subterm corresponding $\bar{0}$, let $\bar{0} \mapsto 0$.

Let

$$\begin{aligned} K^\oplus &= \{(z_{i'}, z_{j'}, z_{k'}) : z_{i'} = z_{j'} \oplus z_{k'}\} \\ K^\neg &= \{(z_{i'}, z_{j'}) : z_{i'} = \neg z_{j'}\} \\ K^{\delta_n} &= \{(z_{i'}, z_{j'}) : z_{i'} = \delta_n z_{j'}\} \\ K^{\bar{0}} &= \{(z_{i'}, 0) : z_{i'} = 0\} \end{aligned}$$

(for each index n occurring in φ), where $z_{i'}, z_{j'}, z_{k'}$ are variables corresponding to either x_φ^i or y^j .

Now, the graph of the operation corresponding to a connective $\langle \oplus, \neg, \{\delta_n\} \rangle$ can be defined by means of a Boolean combination of polynomial equalities and inequalities. Indeed,

²In the original formulation of their logic, Halpern and Pucella allowed real coefficients. However, in order to study computational complexity they needed to have integer coefficients only. Naturally, everything works as well if we have rational coefficients that can be represented as fractions $\frac{a}{b}$ of coprime natural numbers a and b , and whose size $|\frac{a}{b}|$ is given by the sum $|a| + |b|$ of the lengths of a and b written in binary.

$$\begin{aligned}
 s \oplus w = u \quad \text{is} \quad t^\oplus(u, s, w) &:= ((s + w) < 1 \wedge u = (s + w)) \vee ((s + w) \geq 1 \wedge u = 1), \\
 \neg s = w \quad \text{is} \quad t^\neg(w, s) &:= w = 1 - s, \text{ and} \\
 \delta_n s = w \quad \text{is} \quad t^{\delta_n}(w, s) &:= w = \frac{1}{n}s.
 \end{aligned}$$

Now, to each $(z_{i'}, z_{j'}, z_{k'}) \in K^\oplus$ assign $t^\oplus(z_{i'}, z_{j'}, z_{k'})$, to each $(z_{i'}, z_{j'}) \in K^\neg$ assign $t^\neg(z_{i'}, z_{j'})$, to each $(z_{i'}, z_{j'}) \in K^{\delta_n}$ assign $t^{\delta_n}(z_{i'}, z_{j'})$ (for each index n occurring in the connectives δ_n in Φ), and to each $(z_{i'}, 0) \in K^{\bar{0}}$ assign $t^{\bar{0}}(z_{i'}, 0)$, where $t^{\bar{0}}(z_{i'}, 0)$ is $z_{i'} = 0$.

Let a, b, c , and d be the number of occurrences of $\oplus, \neg, \bar{0}$, and all the δ_n , respectively, in φ . Let χ be the following formula:

$$\left(\bigwedge_1^a t^\oplus \right) \wedge \left(\bigwedge_1^b t^\neg \right) \wedge \left(\bigwedge_1^c t^{\bar{0}} \right) \wedge \left(\bigwedge_1^d t^{\delta_n} \right) \wedge (y_m = 1).$$

An easy inspection shows that an assignment of upper probabilities to elementary probabilistic formulas $\pi(\varphi)$ satisfies Φ iff the same assignment to the variables x_φ^i is such that $y_m = 1^3$.

Now, we want to translate χ into a formula in the language of Halpern and Pucella's logic AX^{up} . In χ every formula t^\oplus is a disjunction of two incompatible formulas

$$((s + w) < 1 \wedge u = (s + w)) \quad \text{and} \quad ((s + w) \geq 1 \wedge u = 1).$$

The satisfiability of t^\oplus is then equivalent to the satisfiability of one of the above disjuncts. There are a occurrences of \oplus in Φ , meaning that the satisfiability of Φ is equivalent to the satisfiability of the disjunction of up to 2^a mutually incompatible Boolean conjunctions of linear equations and inequalities. In particular, if an assessment of upper probabilities to elementary probabilistic formulas in Φ satisfies Φ , then the same assignment to the variables x_φ^i satisfies one of the disjuncts in each t^\oplus in χ .

Now, for each occurrence of \oplus , non-deterministically guess if the equation $x \oplus y$ is strictly less than 1 or not. This reduces the satisfiability of χ to the satisfiability of the formula χ' defined as

$$\left(\bigwedge_1^a t'^\oplus \right) \wedge \left(\bigwedge_1^b t^\neg \right) \wedge \left(\bigwedge_1^c t^{\bar{0}} \right) \wedge \left(\bigwedge_1^d t^{\delta_n} \right) \wedge (y_m = 1),$$

where each t'^\oplus corresponds to either

$$((s + w) < 1 \wedge u = (s + w)) \quad \text{or} \quad ((s + w) \geq 1 \wedge u = 1).$$

Now, in χ' there are variables y^i defined by a linear equation. We want to translate χ' into a formula in the language of AX^{up} , so we have to get rid of the variables that do not correspond to elementary probabilistic formulas. Get rid of all the occurrences of the y^j by eliminating all the equations defining the y^j and substituting the definition in the rest of the formula. In this way we reduce χ' to a

³Notice that we are not claiming that χ as a Boolean combination of linear equalities and inequalities is satisfiable iff so is Φ . We are saying that an assessment of upper probabilities satisfies Φ iff the same assessment to the variables x_φ^i satisfies χ .

formula χ'' in which the only variables appearing are the x_φ^i . Substitute the term $\ell(\varphi)$ to each x_φ^i . We eventually obtain a formula in the language of AX^{up} whose satisfiability implies the satisfiability of Φ . Satisfiability in AX^{up} is in NP, hence the claim follows. \blacksquare

Example 4.2 We give an example to show how the above algorithm works. Let Φ be $\neg v(\varphi \rightarrow \psi) \oplus \delta_m v(\psi)$.

$$S = \{v(\varphi \rightarrow \psi), \neg v(\varphi \rightarrow \psi), v(\psi), \delta_m v(\psi), \neg v(\varphi \rightarrow \psi) \oplus \delta_m v(\psi)\}$$

is the set of all subterms of Φ .

Take the following assignment of variables:

$$\begin{aligned} v(\varphi \rightarrow \psi) &\mapsto x_{\varphi \rightarrow \psi}, & \neg v(\varphi \rightarrow \psi) &\mapsto y_1, & v(\psi) &\mapsto x_\psi, \\ \delta_m v(\psi) &\mapsto y_2, & \neg v(\varphi \rightarrow \psi) \oplus \delta_m v(\psi) &\mapsto y_3. \end{aligned}$$

Now, $k^\oplus = \{(y_3, y_2, y_1)\}$, $K^\neg = \{(y_1, x_{\varphi \rightarrow \psi})\}$, and $K^{\delta^m} = \{(y_2, x_\psi)\}$.

Then, we define the formula χ as follows (applying the definition of the graph of the connectives):

$$\begin{aligned} &(y_1 = 1 - x_{\varphi \rightarrow \psi}) \wedge (y_2 = \frac{1}{m} x_\psi) \wedge \\ &(((y_1 + y_2 < 1) \wedge (y_3 = y_1 + y_2)) \vee ((y_1 + y_2 \geq 1) \wedge (y_3 = 1))) \wedge (y_3 = 1). \end{aligned}$$

Now, we have to guess for the occurrence of \oplus which of the disjuncts,

$$((y_1 + y_2 < 1) \wedge (y_3 = y_1 + y_2)) \quad \text{or} \quad ((y_1 + y_2 \geq 1) \wedge (y_3 = 1)),$$

is satisfied. In this particular case, if we want Φ to be satisfiable, we obviously have to choose $((y_1 + y_2 \geq 1) \wedge (y_3 = 1))$.

Then, we get the formula χ'

$$(y_1 = 1 - x_{\varphi \rightarrow \psi}) \wedge (y_2 = \frac{1}{m} x_\psi) \wedge ((y_1 + y_2 \geq 1) \wedge (y_3 = 1)) \wedge (y_3 = 1).$$

Now, to complete the translation, we have to get rid of the variables that do not correspond to elementary probabilistic formulas. We know that $y_1 = 1 - x_{\varphi \rightarrow \psi}$, $y_2 = \frac{1}{m} x_\psi$, and $y_3 = 1$, then we eliminate all those equalities and make the corresponding substitutions, obtaining the formula χ'' :

$$1 - x_{\varphi \rightarrow \psi} + \frac{1}{m} x_\psi \geq 1.$$

Finally, we substitute $\ell(\varphi \rightarrow \psi)$ to $x_{\varphi \rightarrow \psi}$, and $\ell(\psi)$ to x_ψ , obtaining the formula

$$1 - \ell(\varphi \rightarrow \psi) + \frac{1}{m} \ell(\psi) \geq 1,$$

in the language of AX^{up} . The above formula is satisfiable iff there is an upper probability measure such that $\pi(\varphi \rightarrow \psi) + \frac{1}{m} \pi(\psi) \geq 0$. This can be checked in non-deterministic polynomial time in AX^{up} . Then, under the above guess, the satisfiability of $1 - \ell(\varphi \rightarrow \psi) + \frac{1}{m} \ell(\psi) \geq 1$ implies the satisfiability of Φ .

Let Φ be any $\mathcal{U}(\text{RL})$ -formula. We say that Φ is \diamond - r -satisfiable if there exists an Upper Probability Kripke model $\langle W, U, \pi, e \rangle$ such that $e(\Phi) \diamond r$, where $r \in \mathbb{Q} \cap [0, 1]$, and $\diamond \in \{=, \leq, \geq\}$.

Corollary 4.3 *The set of all \diamond - r -satisfiable formulas is in NP.*

Proof. Let r be a rational number in $[0, 1]$ of the form $\frac{n}{m}$. The result immediately follows from the above theorem and from the fact that Φ is \leq - r -satisfiable iff $(\Phi \rightarrow n.\delta_m \bar{1})$ is satisfiable, Φ is \geq - r -satisfiable iff $(n.\delta_m \bar{1} \rightarrow \Phi)$ is satisfiable, and Φ is $=$ - r -satisfiable iff $(\Phi \leftrightarrow n.\delta_m \bar{1})$ is satisfiable. ■

In many real-life situations assessments of uncertainty are not precisely made over a set of events with a specific algebraic structure. Still, such assessments must be required to be coherent, that is: they must satisfy the axioms of a measure whenever they are extended over the whole Boolean algebra generated by those events.

Definition 4.4 Let \mathcal{C} be a countable set of events, and ϕ be a real-valued assessment defined on \mathcal{C} . We call ϕ a coherent upper probability measure if there is an upper probability measure π over the Boolean algebra generated by \mathcal{C} such that $\phi(\varphi) = \pi(\varphi)$ for all $\varphi \in \mathcal{C}$.

Then we have:

Corollary 4.5 *The problem of checking the coherence of a rational assessment of upper probabilities to a finite set of events is in NP.*

Proof. Let $\{\phi(\varphi_i) = \alpha_i\}$, with $\alpha_i \in \mathbb{Q} \cap [0, 1]$, and $1 \leq i \leq n$, be a rational assessment to a finite set of events. Then the coherence of the above assessment is tantamount to checking the simultaneous satisfiability of the formulas $\{v(\varphi_i) \leftrightarrow \alpha_i\}$. This can be clearly translated into a conjunction of formulas in AX^{up} . Hence, the claim follows. ■

5 Final Remarks

Other logical treatments for representing measures of uncertainty within the approach adopted in this paper were presented in several works. We can mention the treatment of probability measures, necessity measures and belief functions proposed by Esteva, Hájek, and Godo in [12, 11, 8, 7]; the treatment of conditional probability proposed by the present author and Godo in [9]; the treatment of (generalized) conditional possibility and necessity given by the present author in [16]; and finally the treatment of simple and conditional non-standard probability given by Flaminio and Montagna in [5]. A more general approach covering fuzzy measures in general was given by the present author in [17].

To conclude let us mention that the main difference between our approach and the one proposed by Halpern and Pucella relies on the properties of the chosen logical framework. Their approach is fundamentally two-valued and is strongly based on the presence of axioms of linear inequalities which allow to represent basic operations between formulas. On the contrary, our approach exploits the advantage that, in Rational Lukasiewicz logic (that is many-valued) the operations associated to the evaluation of the connectives are continuous piecewise linear functions,

whose combinations yield the whole set of continuous piecewise linear polynomial functions with rational coefficients defined over the n th-cube. Therefore, in our treatment we do not need to add axioms for having peculiar operations, since the operations needed to compute with upper probabilities are already available in the semantics of Rational Lukasiewicz logic. From the logical point of view, this allows us to obtain a very elegant and simple treatment .

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