ON THE USAGE OF OPTION CONTRACTS TO MAXIMIZE REVENUES UNDER VALUE INVESTING AXIOMS

Marc Andrés Carcasona

mandresc@uoc.edu

A thesis submitted in partial fulfillment of the requirements for the degree of BSc. in Economics with specialization in "Competitiveness and Growth".

Director Prof. Dr. Nektarios Aslanidis

December 23, 2022
2022/2023 - Q1

On the usage of Option Contracts to Maximize Revenues under Value Investing Axioms

by **Marc Andrés Carcasona** (`mandresc@uoc.edu`)
Universitat Oberta de Catalunya (BSc. in Economics)
December 23, 2022

Abstract

Value investors usually consider that each financial product has an intrinsic value and that the market price will, eventually, tend towards that value. Under this philosophy, investors following this school of thought usually buy assets that are underpriced with respect to their intrinsic values in hopes of selling it in the future at a higher price. This is what is known as a buy and hold strategy. In this thesis, we present an analysis of whether option contracts, a derivative financial product, can be used to increase the revenues obtained under the same market conditions. To do so, we use a series of analytical tools based on the theory of stochastic differential equations and Monte Carlo simulations. We consider different stochastic models for the market such as a linear drift and volatility one, a geometric Brownian motion, a constant elasticity of variance model and a Schwartz's model. To generate Monte Carlo samples the Euler method is employed. The results obtained indicate that, indeed, revenues can be increased with this kind of strategies, yet depending on some hyperparameters of each strategy the uncertainty can also increase or even lead to greater losses than the buy and hold one. Therefore, they should be correctly tuned according to the investors risk-aversion profile. Finally, we apply this strategy in a real market, the S&P500 ETF, to validate that the theoretical results still hold in a more realistic situation. To tune the hyperparameters in this real situation we perform a parameter estimation in a Bayesian framework using a nested sampling algorithm.

Resumen

Los inversores en valor normalmente consideran que cada producto financiero tiene un valor intrínseco y que el precio de mercado tenderá hacia ese valor de forma eventual. Bajo esta filosofía, los inversores compran esos activos que están subvalorados en el mercado con respecto a dicho valor intrínseco esperando que en algún momento el precio de mercado aumente. Esta estrategia es la que comúnmente se conoce como comprar y mantener. En este trabajo presentamos un análisis del posible uso de estrategias basadas en contratos de opciones para aumentar los beneficios obtenidos bajo las mismas hipótesis. Para ello, usamos una serie de herramientas analíticas basadas en la teoría de las ecuaciones diferenciales estocásticas y simulaciones de Monte Carlo. Se consideran diferentes modelos estocásticos para el mercado incluyendo un modelo de volatilidad y *drift* constantes, un movimiento Browniano geométrico, un modelo de elasticidad constante de la varianza y el modelo de Schwartz. Para generar muestras aleatorias en las simulaciones de Monte Carlo se emplea el método de Euler. Los resultados obtenidos indican que, efectivamente, los beneficios se pueden incrementar con este tipo de estrategias, pero tienen una serie de hiperparámetros que definen la estrategia que pueden hacer que la incertidumbre aumente o que incluso se aumenten las pérdidas. Por lo tanto, estos hiperparámetros se tienen que optimizar de acuerdo al perfil y aversión al riesgo de los inversores. Finalmente, aplicamos esta estrategia en un mercado real, el del ETF S&P500, para validar que los resultados teóricos aun se mantienen para una situación más realista. Para poder optimizar los hiperparámetros en este mercado se realiza una inferencia Bayesiana usando un algoritmo de *nested sampling*.

Keywords: value investing, option contracts, stochastic differential equation, Monte Carlo simulation, Bayesian inference, nested sampling.

Palabras clave: inversión de valor, contratos de opciones, ecuaciones diferenciales estocásticas, simulaciones de Monte Carlo, inferencia Bayesiana, *nested sampling*.

ODS: -

Contents

1 — Introduction

There are different ways in which agents typically invest in a financial market. One of the widely accepted approaches is the so-called Value Investing (VI) and is based in finding the intrinsic value of a company using the available information and buying only the stocks that trade at a smaller price in the market than its intrinsic one ([1]–[3]). Therefore, this trading philosophy has a major axiom and is that with time, the market is going to agree with the true value of the stock. This is the major source of revenues from this strategy.

On the other hand, option contracts are another way to trade in the market [4]. They give the option (and thus the name) to buy or sell a given underlying asset, for an agreed fixed price at a given date. The seller always receives a monetary compensation for this contract which is referred as the premium [4]. Depending on how the evolution of the underlying asset price is, this contract will benefit one part or the other. The advantage of this derivative asset is that allows for more flexibility, since it lets the market agents have revenues even if the asset decreases its value. The counterpart is that they usually require a greater expertise to be traded.

In this project we assess the usage of option contracts, instead of the Buy and Hold (BH) approach, to increase the revenues of VI. To do so, the appropriate mathematical and computational tools are going to be applied. Firstly, a modelization of the market in terms of a stochastic process will be done. With this modelization, the statistical properties of some strategies will be analyzed analytically. This first analysis should already be able to answer simple questions about how good are the different strategies with respect to each other. Then, the same set of stochastic equations will be solved numerically to check the validity of the previous results and also to use more advanced strategies or optimize them. Finally, these strategies will be applied to a real market situation to test their applicability in a real situation. In this particular case, the ETF S&P500 will be used [5].

All this procedure will be done assuming the basic axiom of VI which states that with enough time the price of the underlying asset will reach its intrinsic value. This will be translated in some mathematical restrictions that will be discussed later in this work.

1.1 Justification

The major interest in pursuing this project is that VI is a well known, extended and accepted philosophy and has been proven to yield very high returns; any improvement to the trading strategy will generate extra revenues at no more cost. Great part of the work required when VI strategies are applied goes into the actual company valuation. Therefore, adding the extra step of a trading following a strategy based on option contracts should only improve the results almost at no cost.

If good strategies are found, they could be used by hedge funds or private investors that already apply VI strategies to raise their revenues when applying them instead of a traditional BH. Therefore, it could, in principle, benefit any reader that is interested in the topic from an academic perspective but also from a practical point of view, since the impact is potentially high, even if no dominant strategies are found. In that case, it will be mathematically and computationally confirmed that a BH strategy under VI axioms is the best approach.

Furthermore, even if no strictly dominant strategy is found, they will have different expected values and variances, which can represent better strategies depending on the investor's risk-aversion profile. This means more flexibility for future investors when designing their portfolios: they will be able to decide from a variety of market exposure and potential profits. And this is precisely why option contracts are a great product to achieve such richness of combinations that can adapt to every investor's profile and risk-aversion.

1.2 Objectives

Following the aforementioned structure of this work, the main objectives that will serve at the same time as milestones are:

- **Understand and construct a time continuous stochastic model.** To start, a suitable stochastic model needs to be used. It will be important to understand the assumptions and consequences of the choice of the stochastic model that will be used. This can also imply that different ones might be tested to cover a reasonable range of markets.
- **Analytically solve the stochastic models.** After the market model has been chosen, an analytical investigation should be performed. Some models admit analytical solutions and therefore allow for closed form expressions when assessing the different strategies' payoffs. Therefore, in this project this step should be followed as much as possible.
- **Numerically solve the set of equations**. To check the analytical results, numerical Monte Carlo simulations can be performed. Additionally, this computational tool can also be used to test some complicated trading strategies that precludes the analytical treatment. Moreover, this approach allows for an easy optimization of some trading strategies hyperparameters.
- **Test the strategies in a real market.** While the conclusions that might be extracted from the mock market generated by a stochastic model are interesting, there will be unmodeled features that only real markets have. Therefore, the final objective is to test them in a realistic situation.

In a general way, the main objective is to develop a theory and methodology that allows the investigation of new strategies that might potentially increase the revenues of a typical BH strategy. Therefore, making it as generic as possible will be desirable for future inclusions of new market models and/or strategies.

1.3 Literature review

The analysis that is being done is grounded on the same ideas as VI. This philosophy is based on the believe that each asset has an intrinsic or book value which might differ from that at which trades in the market. An important step is then to correctly value an asset and for that there is an extensive literature available (e.g. [1]–[3]). This strategy has proven successful for a variety of investors and hedge funds [1] and, therefore, the literature already points out that it is sensible to try to improve the revenues but starting from this paradigm. Apart from this generic axiom, for the rest of this work, VI will not be further studied, for it is outside of the scope.

Instead, the focus will rely on the mathematical analysis of the market and the different strategies. To this end, mathematical literature on stochastic differential equations will be needed.

For Itô calclculus and stochastic differential equations, some classical works, such as Øksendal's [6], Evans' [7] or Shreve's [8] will be used. These references represent the foundations of all the mathematical tools that will be used throughout this work. On the other hand, options will be studied from the classical perspective of Black and Scholes [9] and Merton [10] and more recent references also use them ([4], [11], [12]). The reason is that to obtain the premium of an option, a model needs to be assumed. Since, again, this is out of the scope of the project the analysis will be kept using the simplest of those but that will already be capable of extracting significant conclusions.

For the Monte Carlo simulations, Refs. [6], [11] will be used. They introduce the subject, describe various techniques to be employed and discuss their pros and cons. Finally, for the Bayesian inference we can use the likelihood proposed in Refs. [12], [13] and for the actual sampling of the posterior, the nested sampling algorithm as described in [14], [15] will be used.

Regarding the specifics being studied (i.e. a strategy using options as a complement to the classical buy and hold from value investing) there is very little scientific and peer-reviewed information available. Some investors have disclosed some options contracts that they have used but in any case there is no rigorous mathematical study analyzing them. The little literature available further justifies this work.

Additionally, a very important decision to be made is regarding the software to use. To this end, there are mainly two main ones that we can use: Mathematica ([16]) and MatLab ([17]). The former is perfectly suited to deal with the analytical calculations that we pretend to carry out and, therefore, will help to extract the statistical properties that we want. The latter can numerically solve stochastic differential equations and, therefore, will be used for Monte Carlo simulations.

1.4 Ethical and social implications

This work, due to its nature, does not treat in any way ethical or social issues. Therefore, it is not applicable a section on this matter.

2 — Mathematical models

For the purpose of analyzing the different strategies, a time continuous representation of the market will be used. In this chapter, the initial mathematical tools that will be used are introduced, then a description of option contracts alongside its valuation is presented, followed by a description of the strategies considered and closed by an explanation of various market models that can be used.

2.1 Initial considerations

A general probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be considered, being Ω a given set, $\mathcal{F}$ a σ-algebra as described in Definition 1 and $\mathbb{P}$ a probability measure as stated in Definition 2.

Definition 1. *A σ-algebra $\mathcal{F}$ is a family of subsets of Ω such that [6]:*

- $\emptyset \in \mathcal{F}$
- $\forall F \in \mathcal{F} \implies \Omega \setminus F \in \mathcal{F}$
- *If* $A_1, A_2, ..., A_n \in \mathcal{F} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{F}$

Definition 2. *A probability measure is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that [6]:*

- $\mathbb{P}(\emptyset) = 0,\ \mathbb{P}(\Omega) = 1$
- *If $A_1, A_2, ..., A_n \in \mathcal{F}$ with $A_i \cap A_j = \emptyset,\ \forall i \neq j$ then*

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

In general, the value of a stock at time t will be denoted by S_t and it will be governed by the following generic Stochastic Differential Equation (SDE) [18]

$$\mathrm{d}S_t = \mu(t, S_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t \tag{2.1}$$

where $\mu : [0, T] \times \Omega \to \mathbb{R}$ denotes the drift of the process, $\sigma : [0, T] \times \Omega \to \mathbb{R}$ the variance and W_t is a Wiener process as described in Definition 3. The initial condition will be denoted as $S_{t=0} \equiv S_0$.

Definition 3. *We define by Wiener process any continuous stochastic process W_t, $\forall t \in [0, T]$ such that:*

- $W_0 = 0$.
- $W_t - W_s$ *is independent of* $W_{t'} - W_{s'}$ *for* $0 \leq s' < t' \leq s < t \leq T$.
- $W_t - W_s \sim \mathcal{N}(0, t-s)$.

It will be useful to know some important properties for this random process. The pdf of W_t *is*

$$p_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} \ , \tag{2.2}$$

which implies $\mathbb{E}[W_t] = 0$, $\mathbb{V}\mathrm{ar}[W_t] = t$ *and* $K(t,s) = \mathrm{cov}(W_t, W_s) = \min(t,s)$.

Equation (2.1) is suitable to be represented as an Itô integral as

$$S_t = S_0 + \int_0^t \mu(s, S_s)\mathrm{d}s + \int_0^t \sigma(s, S_s)\mathrm{d}W_s \ . \tag{2.3}$$

This representation of the SDE allows for a study of the statistical properties of the stochastic process as long as these integrals can be solved. Itô's lemma (defined in Lemma 1) will be used to solve these integrals. This lemma is based itself on the following Taylor expansion of the differential of a stochastic function

$$\mathrm{d}f(x) = f'(x)\mathrm{d}x + \frac{1}{2}f''(x)\mathrm{d}x^2 + \mathcal{O}(\mathrm{d}x^3) \ . \tag{2.4}$$

Additionally, the following assumptions are also commonly made [6]

$$\mathrm{d}t^2 = 0, \qquad \mathrm{d}t\mathrm{d}W_t = 0, \qquad \mathrm{d}W_t^2 = \mathrm{d}t \tag{2.5}$$

Lemma 1. *Let* $f(t,x) \in \mathcal{C}^2$ *and a stochastic process* $\mathrm{d}S_t = \mu(t, S_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t$, *then*

$$\mathrm{d}f(t,x) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \sigma\frac{\partial f}{\partial x}\mathrm{d}W_t \tag{2.6}$$

To integrate a function $g(t, W_t) \in \mathcal{C}^2$ with respect to a Wiener process

$$I = \int_0^t g(s, W_s)\mathrm{d}W_s \ , \tag{2.7}$$

one can use Itô's lemma, yielding

$$\int_0^t g(s, W_s)\mathrm{d}W_s = f(t, W_t) - f(0,0) - \int_0^t \frac{\partial f}{\partial t}(s, W_s) + \frac{1}{2}\frac{\partial g}{\partial x}(s, W_s)\mathrm{d}s \ , \tag{2.8}$$

where $f(t,x)$ is the antiderivative of $g(t,x)$ with respect to the second argument. This is $\frac{\partial f}{\partial x} = g(t,x)$. This would reduce the problem to solving integrals of the form

$$I = \int_a^b f(s, S_s)\mathrm{d}s \ . \tag{2.9}$$

While this is a random variable in general, we might compute some relevant quantities, such as the expected value or the variance. To do so we need to use two important properties. The first one is that the expected value of the integral will equal the integral of the expected value of the integrand. This is

$$\mathbb{E}\left[\int_a^b S_t \mathrm{d}t\right] = \int_a^b \mathbb{E}\left[S_t\right] \mathrm{d}t \ . \tag{2.10}$$

Similarly, to estimate the variance of an integral with a stochastic process as integrand, we will rely on the autocovariance function $K(t,s)$ as

$$\mathbb{V}\mathrm{ar}\left[\int_a^b S_t \mathrm{d}t\right] = \int_a^b \int_a^b K(t,s) \mathrm{d}t\mathrm{d}s \tag{2.11}$$

where the covariance function is defined as

$$K(t,s) = \mathbb{E}\left[S_t, S_s\right] - \mathbb{E}\left[S_t\right]\mathbb{E}\left[S_s\right] \tag{2.12}$$

At this stage is also important to study the existence and uniqueness of the original SDE as defined in Eq. (2.1). Applying Theorem 1, the conditions that the SDE must fulfill are Eqs. (2.13) and (2.14) and, therefore, we the different models of the market are presented they should be checked to assess if the problem is well-posed.

Theorem 1. ***Existence and uniqueness of solution for stochastic differential equations*** *([6]) Let $T > 0$ and $\mu : [0,T] \times \Omega \to \mathbb{R}$ and $\sigma : [0,T] \times \Omega \to \mathbb{R}$ be measurable functions for which (linear growth assumption)*

$$|\mu(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \quad x \in \Omega, t \in [0,T] \tag{2.13}$$

for some arbitrary constant C and where $|\sigma|^2 = \sum |\sigma_{i,j}|^2$ and which additionally satisfy that (global Lipschitz assumption)

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|, \quad x, y \in \Omega, t \in [0,T] \tag{2.14}$$

for some constant D. Let also Z be a random variable, independent of the σ-algebra generated by $W_s(\cdot), s \geq 0$ and such that $\mathbb{E}\left[|Z|^2\right] < \infty$. Them the stochastic differential equation as defined in Eq. (2.1) has a unique $t-$continuous solution $S_t(\omega)$ that is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $W_s(\cdot)$ with $s \leq t$ and

$$\mathbb{E}\left[\int_0^T |S_t|^2 \mathrm{d}t\right] < \infty \ . \tag{2.15}$$

2.2 Option contracts

Options are defined as contracts between a buyer and a seller for which the former has the right, yet no the obligation, to buy or sell a specific quantity of an underlying asset. Moreover, to acquire this right it pays a quantity of money, called the premium, to the seller of the contract who, in its turn, is obliged to execute the other part of the deal in case the holder was willing to exercise it [4]. The buying or selling of this underlying asset will be done at a specified price, called the strike price, on or before a specified date, called the maturity of the option. These type of contracts are considered to be derivative financial products as they depend on the price of an underlying asset, yet not in a linear way. Moreover, it is important to mention that there are different types of options and each one of them will have a different kind of behavior. For example, American options can be exercised at any time between the contract signing and the maturity date, while European options can only be exercised at the specified date [4]. Similarly, other exotic options exist that can introduce different payoffs.

Apart from modeling the market, a proper model of the option contracts will be needed. The theory that will be employed in this case is the basic and classical theory by Black, Scholes and Merton ([9], [10]). The basic formalism assumes a market that follows a geometric Brownian motion as

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \ , \tag{2.16}$$

and a risk free, non-stochastic, bond that follows [18]

$$\mathrm{d}B_t = r B_t \mathrm{d}t \ , \tag{2.17}$$

being r the risk-free interest rate.

The question is then to determine the value of an option contract that we will be denoted by V and that will be a stochastic variable as it will depend on the value of the asset, which is stochastic in nature. One can apply Itô's lemma to get to ([9], [18], [19])

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \ , \tag{2.18}$$

where σ is the volatility of the underlying asset.

Equation (2.18) is a Prtial Differential Equation (PDE) that will be subject to boundary and initial conditions in order to be fully specified. These will depend on the type of contract and only some of them will be considered throughout this project. For example, a vanilla European call option would have the following final condition (i.e. specified at $t = T$)

$$V(t = T, S) = \max(S - K, 0) \ , \tag{2.19}$$

and the boundary conditions

$$\lim_{S\to 0} V(t, S) = 0 \qquad \lim_{S\to\infty} V(t, S) = S \ . \tag{2.20}$$

Assuming constant volatility, Eq. (2.18) can be integrated analytically for some simple cases. For example, for a European vanilla call option, this is [19]

$$C(t, S_t|K, \sigma, r, T) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \ , \tag{2.21}$$

with $N(x)$ being the standard normal distribution,

$$d_1 = \frac{\log(S_t/K) + \left(r + \sigma^2/2\right)(T - t)}{\sigma\sqrt{T - t}} \tag{2.22}$$

and

$$d_2 = \frac{\log(S_t/K) + \left(r - \sigma^2/2\right)(T - t)}{\sigma\sqrt{T - t}} \ . \tag{2.23}$$

Similarly, for a put European option, this is

$$P(t, S_t|K, \sigma, r, T) = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1) \ . \tag{2.24}$$

This valuation is important because will determine the premium of the option contract. Therefore, selling a call or put will imply collecting the premium while buying any of these contracts will mean paying upfront the premium.

2.3 Proposed strategies

We will define the payoff of a strategy as a function $\xi : [0,T] \times \Omega \to \mathbb{R}$ that in the most general case will depend on the value of the asset and, therefore, will be stochastic in nature.

Buy and hold

The easiest and most common, yet nonetheless, a strategy anyway, is that of buy and hold. It consists on buying an asset at a given price and selling it at its intrinsic value. Therefore, the payoff is going to be linearly related to the price of the stock. The payoff function for this strategy is then

$$\xi(t) = S_t - S_0 \ , \tag{2.25}$$

which solution coincides with that of Eq. (2.1). This already indicates that the expected value and variance are those of the underlying asset implying the same expected returns and volatility. This strategy, albeit simple, has reported great revenues to big hedge funds and private investors [1]. Therefore, we will compare the other strategies with this one, since in case of no clear improvement, a buy and hold will usually be safer, as experience has shown.

Selling puts while waiting to sell the asset (strategy 1)

Assuming that the price of the asset is going to reach the intrinsic valuation of such asset is a core aspect of the VI philosophy. But, while this happens, there is time in between in which under the traditional BH strategy one can only wait. The question here is: what if in the meantime we sell puts to increase our revenues? Since we are assuming that the price will go up, we should always collect the premium.

We first need to define the maturity of the options. There is no easy way of determining it because of the stochastic nature of the market. Nonetheless, we can get an educated guess of a reasonable value. We will assume constant drift and volatility and since we know that at a time T the PDF of the stochastic process is going to be

$$p_{S_T}(S) = \frac{1}{\sqrt{2\pi T}\sigma} \exp\left\{ -\frac{(S - T\mu)^2}{2T\sigma^2} \right\} \ , \tag{2.26}$$

then the CDF will be

$$F_{S_T}(S) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{S - T\mu}{\sqrt{2T}\sigma}\right)\right] \ . \tag{2.27}$$

We can determine the time T_α at which with a α confidence level, the price will not have yet arrived at S_v (the valuation price). In this way we ensure ourselves that when the contract expires the price of the asset will still have not reached it's intrinsic value. Solving the equation $F_{S_T}(\Delta \mathrm{S} = S_v - S_0) = \alpha$ yields the maturity

$$T_\alpha = \frac{2\Delta \mathrm{S}\mu + 2\theta_\alpha^2\sigma^2}{2\mu^2} \pm \sqrt{\frac{2\Delta \mathrm{S}\theta_\alpha^2\mu\sigma^2 + \theta_\alpha^4\sigma^4}{\mu^4}} \ , \tag{2.28}$$

where we have defined $\theta_\alpha = \mathrm{erf}^{-1}(2\alpha - 1)$ and typically $\alpha = 0.05$. Under reasonable parameters the positive solution will be the one to employ.

Then, we can consider the situation where we acquire the stock and where we sell N puts. The payoff in this case will be

$$\xi(t) = (S_t - S_0) + N \min[S_t - K, P(t = 0, S_0|K, \sigma, r, T_\alpha)]\Theta(t - T_\alpha) \ , \tag{2.29}$$

where $\Theta(x)$ is the Heaviside step function. It becomes clear that this strategy differs from the buy and hold one by the second term. Assuming that $t > T_\alpha$ (which by construction should be the predominant case) we can express it as

$$\xi(t) = \begin{cases} S_t(N+1) - S_0 - NK & \text{if } S_t - K \leq P \\ S_t - S_0 + NP & \text{if } S_t - K > P \end{cases} , \tag{2.30}$$

which can give a little bit of more insight. If we plot this function for a given set of parameters (for now completely random but reasonable) the result is the one displayed in Fig. 2.1. The first image shows the payoff obtained in the classical buy and hold strategy. The result only depends on S_t and in a linear way. On the other hand, if we consider that we sell one put, the payoff function does strongly depend on both the execution price (K) and S_t. Although the maximum payoff increases a little bit for this region of the parameter space, in general there is more possibilities of loosing if compared to the previous buy and hold strategy. Specially when the execution price is big, the chances of loosing more money increase. When we consider more contracts, the winning region decreases but the possible revenues increase. There are again two very differentiated regions and correspond to where the jump of the payoff functions occurs.

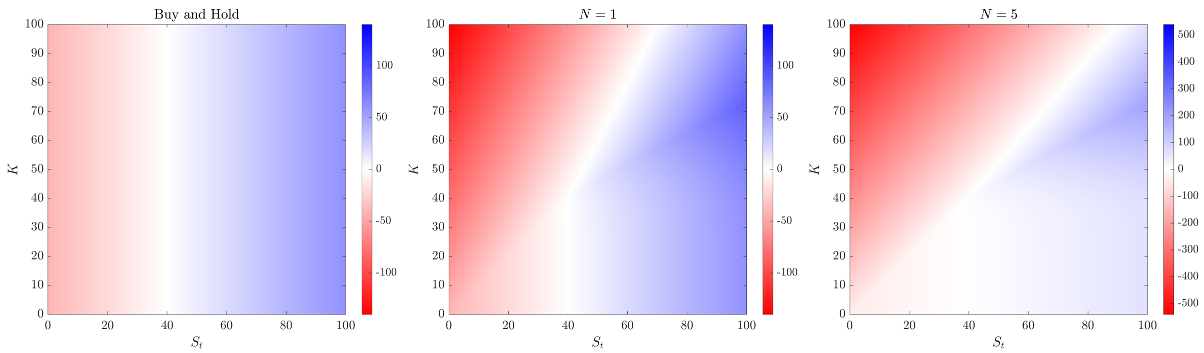

Figure 2.1: Example of the payoff obtained for a simple buy and hold strategy, selling $N = 1$ and selling $N = 10$ puts. The rest of parameters are: $\sigma = 0.1$, $S_v = 60$, $S_0 = 40$, $\mu = 5$ and $r = 0.02$.

Although this seems the holy grail it is of great importance pointing out that the number of contracts that can be signed is not infinite. In the rare, yet not impossible event, that the stock price dropped, the amount of loses can be big, and the seller of the put should have the money ready to make the payment. This is why in practice, N will be limited by the budget.

Selling calls while waiting to buy the asset (strategy 2)

This strategy has a complete different premise. We now consider the case where our valuation is lower than the current market or traded price. Therefore, under the VI paradigm, we shouldn't do anything, for it is overpriced and only wait until this price is low enough to buy it. This means losing a precious time that we could be using to obtain revenues and this is where this strategy might be used.

While we are waiting for the price to come down we can sell puts as well. In this way, if the price drops to or below the one that we wanted to buy the stock we will buy it and that's already good, because we were already comfortable with paying this amount to acquire the stock, but if the price doesn't drop we will be able to keep the premium.

In this case the payoff is very simple because it directly coincides with that of the puts

$$\xi(t) = N \min[S_t - K, P(t=0, S_0|K, \sigma, r, T_\alpha)]\Theta(t - T_\alpha) \ , \tag{2.31}$$

In this case, though, we are considering that the market has a negative drift (i.e. $\mu < 0$) as would do under the VI axioms. This strategy yields a similar, yet different result than the previous one as can be seen in Fig 2.2.

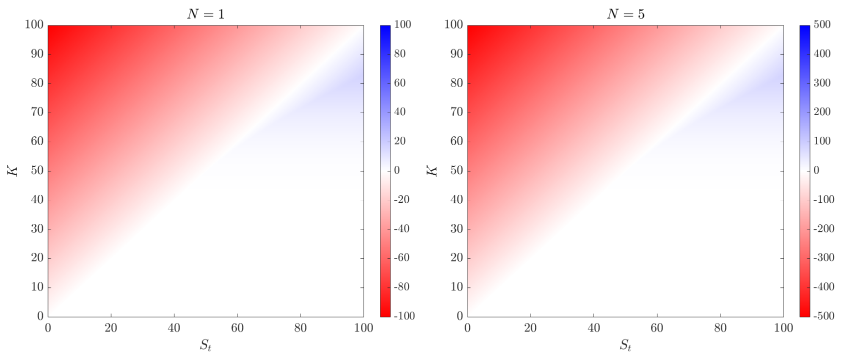

Figure 2.2: Example of the payoff obtained selling $N = 1$ and $N = 10$ puts while waiting the price to go down. The rest of parameters are: $\sigma = 0.1$, $S_v = 40$, $S_0 = 60$, $\mu = -5$ and $r = 0.02$.

There are two very differentiated regions in this case divided by the line $S_t = K$. When $S_t > K$ there are positive revenues, whilst for $S_t < K$ there are losses. There is a clear asymmetry in the amounts that can be won and the ones that can be lost. As expected, the amount of possible losses surpasses that of the revenues. In this case, the effect of increasing the number of contracts is completely linear.

2.4 Market models

We have introduced in Eq. (2.1) a general stochastic equation. We can now consider some particular cases of the form of the functions $\mu(t, S_t)$ and $\sigma(t, S_t)$ that are able to capture the dynamics of the market.

The simplest, yet very interesting, case is that of considering that both are constants, i.e. $\mu(t, S_t) = \mu$, $\mu \in \mathbb{R}$ and $\sigma(t, S_t) = \sigma$, $\sigma \in \mathbb{R}$. This case is general enough to accommodate a wide variety of process and, therefore, will always be the first one that will be used. As stated before, it is important to check whether this SDE actually has a solution. The fist condition that must be fulfilled is that of linear growth and it can be easily checked that indeed

$$|\mu| + |\sigma| \leq C(1 + |x|) \ ,$$

as for example $C = |\mu| + |\sigma|$ already satisfies this condition. The global Lipschitz condition is also satisfied since

$$|\mu - \mu| + |\sigma - \sigma| = 0 \leq D|x - y| \ .$$

We can find the solution to this SDE by simply evaluating Eq. (2.3). This inmediately yields

$$S_t = S_0 + \mu \int_0^t \mathrm{d}t + \sigma \int_o^t \mathrm{d}W_t = S_0 + \mu t + \sigma W_t \ , \tag{2.32}$$

From this solution it easy to compute certain basic properties of this market. The first of them is that the expected value will be

$$\mathbb{E}\left[S_t\right] = \mathbb{E}\left[S_0 + \mu t + \sigma W_t\right] = S_0 + \mu t \ , \tag{2.33}$$

while its variance

$$\mathbb{V}\mathrm{ar}\left[S_t\right] = \mathbb{V}\mathrm{ar}\left[\sigma W_t\right] = \sigma^2 \mathbb{V}\mathrm{ar}\left[W_t\right] = \sigma^2 t \ . \tag{2.34}$$

Finally, the pdf of the price at a time t will be

$$p_{S_t}(S) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{(S - S_0 - \mu t)^2}{2\sigma^2 t}\right\} \tag{2.35}$$

Another very important model found in the literature is that of the geometric Brownian motion ([9], [10]) as has been introduced with the option contracts. In that case, the function of the drift is $\mu(t, S_t) = \mu S_t,\ \mu \in \mathbb{R}$ and the one for the volatility, $\sigma(t, S_t) = \sigma S_t,\ \sigma \in \mathbb{R}$. This one produces an exponential growth.

The conditions of linear growth in this case is verified since

$$|\mu S_t| + |\sigma S_t| \leq C(1 + S_t)$$
$$(\mu + \sigma) S_t \leq C(1 + S_t)$$

and for example with $C = \mu + \sigma$ this condition is automatically verified. On the other hand, the global Lipschitz condition states that

$$|\mu S_t - \mu S_s| + |\sigma S_t - \sigma S_s| \leq D|S_t - S_s|$$
$$\mu(S_t - S_s) + \sigma(S_t - S_s) \leq D|S_t - S_s|$$

which again is easily satisfied for $D = \mu + \sigma$.

To integrate this SDE the following relation simplifies the analysis

$$\mathrm{d}\ln S_t = \frac{\mathrm{d}S_t}{S_t} - \frac{(\mathrm{d}S_t)^2}{2S_t^2} = \frac{\mathrm{d}S_t}{S_t} - \frac{\sigma^2}{2}\mathrm{d}t \ , \tag{2.36}$$

since plugging now the definition of $\mathrm{d}S_t$ for this SDE yields,

$$\mathrm{d}\ln S_t = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}W_t \ . \tag{2.37}$$

This is now exactly as the previous linear SDE only that solving for $\ln S_t$ instead of S_t. It is immediate to see that the solution sought is

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\} \ . \tag{2.38}$$

This SDE has an expected value of

$$\mathbb{E}\left[S_t\right] = S_0 e^{t\mu} \ , \tag{2.39}$$

a variance of

$$\mathbb{V}\text{ar}\left[S_t\right] = S_0^2 e^{2t\mu}\left(e^{t\sigma^2} - 1\right) \ , \tag{2.40}$$

and the pdf at any given time of

$$p_{S_t}(S) = \frac{1}{S\sqrt{2\pi t\sigma^2}} \exp\left\{-\frac{\left(\ln S_t - \ln S_0 - t\mu + \frac{t\sigma^2}{2}\right)^2}{2t\sigma^2}\right\} \tag{2.41}$$

A variannt of this model is what is known as Constant Elasticity of Variance model (CEV) and changes the volatility function to $\sigma(t, S_t) = \sigma S_t^{\gamma},\ \sigma, \gamma \in \mathbb{R}$ [11]. Despite being a small modification, it has a huge impact, since now this SDE doesn't have a closed analytical solution. The only quantity that can be estimated is the expected value and is the same as in the geometric Brownian motion case. Therefore, we will need to use the Monte Carlo techniques to obtain various paths and compute the desired properties of this market.

We can finally consider a process of the form

$$\mathrm{d}S_t = \gamma[\mu - \ln(S_t)]S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \tag{2.42}$$

as has previously been successfully used to model commodities and other markets [20], [21]. We will call it Schwartz's model as it was introduced in Ref. [20]. This process can be converted into the widely known Ornstein-Uhlenbeck process by transforming it into the variable $X_t = \ln(S_t)$. To prove this we perform this change of variable to obtain

$$\mathrm{d}S_t = \gamma\left[\mu - X_t\right] e^{X_t}\mathrm{d}t + \sigma e^{X_t}\mathrm{d}W_t \ . \tag{2.43}$$

At the same time, applying Itô's lemma,

$$\mathrm{d}S_t = \mathrm{d}\left(e^{X_t}\right) = e^{X_t}\mathrm{d}X_t + \frac{1}{2}e^{X_t}\mathrm{d}X_t^2 \ , \tag{2.44}$$

which combined with Eq. (2.43) leads to

$$\mathrm{d}X_t + \frac{\mathrm{d}X_t^2}{2} = \gamma\left[\mu - X_t\right]\mathrm{d}t + \sigma\mathrm{d}W_t \ . \tag{2.45}$$

To obtain the value of $\mathrm{d}X_t^2$ we can consider Itô's lemma but for the inverse transformation. The result is

$$\mathrm{d}X_t = \mathrm{d}(\ln S_t) = \frac{\mathrm{d}S_t}{S_t} - \frac{1}{2S_t^2}\mathrm{d}S_t^2 \ , \tag{2.46}$$

which directly allows to square it to find the sought value equaling

$$\mathrm{d}X_t^2 = \frac{\mathrm{d}S_t^2}{S_t^2} + \mathcal{O}\left(\mathrm{d}S_t^3\right) \ . \tag{2.47}$$

Finally, the value of $\mathrm{d}S_t^2$ can be obtained by directly squaring both sides of Eq. (2.42), yielding

$$\mathrm{d}S_t^2 = \gamma^2[\mu - \ln S_t]^2 S_t^2 \mathrm{d}t^2 + \gamma[\mu - \ln S_t]S_t^2\sigma\mathrm{d}t\mathrm{d}W_t + \sigma^2 S_t^2 \mathrm{d}W_t^2 = \sigma^2 S_t^2 \mathrm{d}t \ , \tag{2.48}$$

and where Itô's assumptions (relations described in Eq. (2.5)) have been used. This implies that

$$\mathrm{d}X_t^2 = \sigma^2 \mathrm{d}t \ , \tag{2.49}$$

and that the stochastic process that follows the new variable is

$$\mathrm{d}X_t = \gamma[\kappa - X_t]\mathrm{d}t + \sigma \mathrm{d}W_t \ , \tag{2.50}$$

with

$$\kappa = \mu - \frac{\sigma^2}{2\gamma} \ . \tag{2.51}$$

Equation (2.50) is indeed the so-called Ornstein-Uhlenbeck SDE and can be integrated. We define a new variable $Y_t = X_t - \kappa$, which modifies the SDE to

$$\mathrm{d}Y_t = -\gamma Y_t \mathrm{d}t + \sigma \mathrm{d}W_t \ . \tag{2.52}$$

Now, introducing yet a new variable of the form $Z_t = e^{\gamma t} Y_t$ and noting that

$$\mathrm{d}Z_t = \gamma e^{\gamma t} Y_t \mathrm{d}t + e^{\gamma t} \mathrm{d}Y_t = \sigma e^{\gamma t} \mathrm{d}W_t \ , \tag{2.53}$$

we can explicitly integrate this SDE to get

$$Z_t = Z_0 + \sigma \int_0^t e^{ut} \mathrm{d}W_u \ . \tag{2.54}$$

Undoing the changes of variables we arrive to the solution

$$X_t = \kappa + e^{-\gamma t}(X_0 - \kappa) + \sigma \int_0^t e^{ut} \mathrm{d}W_u \ . \tag{2.55}$$

The original random process has then a mean of

$$\mathbb{E}\left[S_t\right] = \exp\left(e^{-\gamma t} \ln S_0 + \frac{\sigma^2\left(1 - e^{-2\gamma t}\right)}{4\gamma} + \kappa e^{-\gamma t}\left(e^{\gamma t} - 1\right)\right) \ , \tag{2.56}$$

and a variance

$$\mathbb{V}\mathrm{ar}\left[S_t\right] = \frac{1}{2\gamma e^{2\gamma t}}\left(e^{\frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)} - 1\right) \exp\left(-\sigma^2 + e^{2\gamma t}\left(4\gamma\kappa + \sigma^2\right) + 4\gamma e^{\gamma t}(\ln S_0 - \kappa)\right) \ . \tag{2.57}$$

Finally, the pdf at time t of this process can be found and equals

$$p_{S_t}(S) = \sqrt{\frac{\gamma e^{2\gamma t}}{S^2 \pi \sigma^2 (e^{2\gamma t} - 1)}} \exp\left(-\frac{\gamma e^{2\gamma t}\left(\ln S - e^{-\gamma t}\kappa\left[\ln S_0 + e^{\gamma t} - 1\right]\right)^2}{\sigma^2\left(e^{2\gamma t} - 1\right)}\right) \ . \tag{2.58}$$

With these four SDEs there is a representative enough selection of models that can be used to study the different strategies. Depending on the market characteristics, the strategies might yield a different outcome. Therefore, it will be important in a real application that the correct market model is employed. This is why in Chapter 4 the Bayesian inference is introduced and a proper way to decide which one fits best the market data is described.

3 — Monte Carlo simulations

In this chapter, the Monte Carlo simulations will be described and the main results obtained presented. This step is extremely important since some market models do not admit an analytic solution and also because some strategies are hard or impossible to be treated analytically. Therefore, being able to generate sample paths of the stochastic process and analyzing the statistics associated to them is the most generic way to perform the study.

This chapter contains three sections. The first one will introduce the methods that can be used to generate Monte Carlo sample paths of a SDE. The second one will display some results obtained by applying such methods. The third one discusses how a strategy can be optimized to match some investors criteria.

3.1 Generation of samples

First of all, in this section the methods to simulate a generic SDE will be presented. An important concept to be taken into account is that of the order of convergence. Intuitively, this is how fast the error between the numerical approximation and the true solution tends to zero with the refinment of the discretization. In the case of the SDEs we can distinguish between the strong and weak order of convergence. They are defined in Definition 4 and Definition 5, respectively.

Definition 4. *(as stated in Ref. [11]) The time-discrete approximation X_δ of the continuous time random process X, being δ the maximum time increment, is said to have a γ strong convergence rate if for any given time horizon T then*

$$\mathbb{E}\left[|X_\delta(T) - X(T)|\right] \leq C\delta^\gamma, \qquad \forall \delta < \delta_0 \ , \tag{3.1}$$

being δ_0 and C constants.

Definition 5. *(as stated in Ref. [11]) The time-discrete approximation X_δ of the continuous time random process X, being δ the maximum time increment, is said to have a β weak convergence rate if for any given time horizon T and any $2(\beta+1)$ continuous differentiable function g then*

$$|\mathbb{E}_g[X_\delta(T)] - \mathbb{E}_g[X(T)]| \leq C\delta^\beta, \qquad \forall \delta < \delta_0 \ , \tag{3.2}$$

being δ_0 and C constants.

For each numerical scheme considered, then, its convergence can be computed. Usually, faster convergence is desired but it is not the only characteristic to be taken into account. In general, other properties of the numerical scheme have to be considered, since they can be as important as the convergence. One of such is the stability of the numerical scheme, as the solution for typical problems must be kept bounded and finite. Equally important is the computation time. Higher order schemes, although yielding a faster convergence, can have slower computation times. This usually is related to the numerical complexity, which increases for such schemes. Therefore, a good balance between fast rate of convergence and computational time must be reached.

Some of the numerical schemes that can be used are:

- **Euler scheme**. Considering the general SDE introduced in Eq. (2.1) the Euler approximation to generate the $i+1$ sample is defined as

$$S_{i+1} = S_i + \mu(t_i, S_i)(t_{i+1} - t_i) + \sigma(t_i, S_i)(W_{i+1} - W_i)\ . \tag{3.3}$$

It becomes clear that this scheme is an explicit one, as at each time step only the previous state is needed. Similarly, the only needed ingredient is that of generating an increment of the Wiener process, but according to its properties described in Definition 3, this can be computed as

$$W(t + \Delta t) - W(t) \sim \mathcal{N}(0, \Delta t) \sim \sqrt{\Delta t}\mathcal{N}(0, 1)\ . \tag{3.4}$$

This numerical scheme has a strong convergence rate of $\gamma = 1/2$ [11], weak convergence $\beta = 1$ and can be implemented very easily. At the same time, it is extremely fast as allows for a vectorization and parallelization in a straight forward manner. There is no iteration needed at each time step, only the sampling from a standard normal distribution, which is already efficiently implemented in the vast majority of modern computer languages.

- **Milstein scheme**. This approach slightly differs from Euler's scheme by adding a second order term. Using Itô's lemma, the approximation of S_{i+1} can be computed as [11]

$$\begin{aligned} S_{i+1} =& S_i + \mu(t_i, S_i)(t_{i+1} - t_i) + \sigma(t_i, S_i)(W_{i+1} - W_i) \\ &+ \frac{1}{2}\sigma(t_i, S_i)\frac{\partial \sigma}{\partial x}(t_i, S_i)\left\{(W_{i+1} - W_i)^2 - (t_{i+1} - t_i)\right\}\ . \end{aligned} \tag{3.5}$$

The advantage of this other scheme is that it possess a faster converge rate. More precisely, the strong and weak convergence rates are $\gamma = \beta = 1$. As long as the derivative of the volatility function can be found, this approach will yield a better convergence. As it happened with the Euler scheme, this one is completely explicit and only requires to draw random numbers from a standard normal distribution. Therefore, it is also easily implemented in any programming language and will be fast to evaluate.

- **Predictor-corrector scheme**. This family of algorithms have usually two steps. The first one is trying to predict the future value by using a simple Euler iteration. This means that the predicted value at the future time is

$$\tilde{S}_{i+1} = S_i + \mu(t_i, S_i)\Delta t + \sigma(t_i, S_i)\sqrt{\Delta t}Z\ , \tag{3.6}$$

where $Z \sim \mathcal{N}(0, 1)$. Then, the final estimation is the corrected version of the predicted step and it is computed as [11]

$$\begin{aligned} S_{i+1} =& S_i + \left(\alpha\tilde{\mu}(t_{i+1}, \tilde{S}_{i+1}) + (1 - \alpha)\tilde{\mu}(t_i, S_i)\right)\Delta t \\ &+ \left(\eta\sigma(t_{i+1}, \tilde{S}_{i+1}) + (1 - \eta)\sigma(t_i, S_i)\right)\sqrt{\Delta t}Z\ , \end{aligned} \tag{3.7}$$

being

$$\tilde{\mu}(t_i, S_i) = \mu(t_i, S_i) - \eta\sigma(t_i, S_i)\frac{\partial\sigma}{\partial x}(t_i, S_i) \ . \tag{3.8}$$

The two parameters $\alpha \in [0, 1]$ and $\eta \in [0, 1]$ control the predictor-corrector behavior. For example, setting both of them to 0 one recovers the Euler method.

Any of these schemes can generate samples from the different market models considered and with enough accuracy. After the samples are drawn, the pdf of $S(T)$ can be estimated by means of the kernel density estimator. This is computed as [22]

$$\hat{f}_h(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right) \ , \tag{3.9}$$

where $(x_1, x_2, ..., x_i, ..., x_n)$ are independent and identically distributed samples drawn from the distribution, $h > 0$ is usually called the bandwidth and controls the smoothing of the density estimation and $K(x)$ is the kernel function. For the present analysis, the standard normal distribution is used for the latter. This implies

$$K(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\} \ . \tag{3.10}$$

The bandwidth h will be tuned manually to observe a smooth enough distribution and to start iterating an optimal value will be used. Defining the mean integrated squared error as [23]

$$\mathrm{MISE}(h) = \mathbb{E}\left[\int \left(\hat{f}_f(x) - f(x)\right)^2 \mathrm{d}x\right] \ , \tag{3.11}$$

and under weak assumptions on the pdf and the kernel employed, one can find that the minimum of the asymptotic of the MISE is found at [23]

$$h = Cn^{-1/5} \ , \tag{3.12}$$

where C is a constant that depends on the unknown density and its derivatives. As a rule of thumb and assuming that the underlying distribution is Gaussian, the optimal bandwidth is [23]

$$h = \left(\frac{4\hat{\sigma}}{3n}\right)^{1/5} \ , \tag{3.13}$$

being $\hat{\sigma}$ the estimation of the width of the underlying Gaussian distribution. This value is the one used to start iterating to find a good smoothing parameter for the kernels displayed in this work.

3.2 Application to the market models

With any of the methods discussed in Sec. 3.1 we can generate a set of sample paths for the different market models considered. Due to its simplicity in the implementation and the fast sampling speed, the Euler method i going to be employed.

First of all, we can simulate each one of the markets and compare some of them with the theoretical pdfs that are expected. The first of them is the lienar SDE model. For a maturity of

$T = 10$, $n = 100$ steps (i.e. $\Delta t = 0.1$), $\mu = 5$, $\sigma = 10$ and $S_0 = 40$ the result is that displayed in Fig. 3.1. This one, since has a simple pdf is easy to cross-check and indeed the results match up to a reasonable degree. This means that for this model we are able to generate reliable samples.

The next one to be tested is that of the geometric Brownian motion. For a maturity of $T = 0.3$, $n = 100$ steps (i.e. $\Delta t = 0.003$), $\mu = 5S_t$, $\sigma = S_t/2$ and $S_0 = 40$ the paths are displayed in Fig. 3.2. In this case the pdf obtained is the characteristic log-normal distribution expected for this stochastic process. It also matches very well with the theoretical prediction derived.

For the CEV model with the same time parameters as the Brownian motion but with $\mu = 5S_t$, $\sigma = S_t/2$ and $S_0 = 40$ and $\gamma = 0.6$ we obtain the sample paths shown in Fig. 3.3. In this case it is worth mentioning that since the specific choice of γ has been smaller than 1 this has reduced the uncertainty and the obtained distribution resembles more that of a Gaussian and the dispersion between each path has decreased, as it has to be expected by the form of the SDE.

Finally, in the case of Schwartz's model, using the same time parameters as for the Brownian and CEV models, $\mu = 2(5 - \ln S_t)S_t$, $\sigma = S_t$ and $S_0 = 40$, we obtain the results displayed in Fig. 3.4. All in all, the method of generating samples via the Euler scheme works as intended for all the different market models that are considered trhoughout this work.

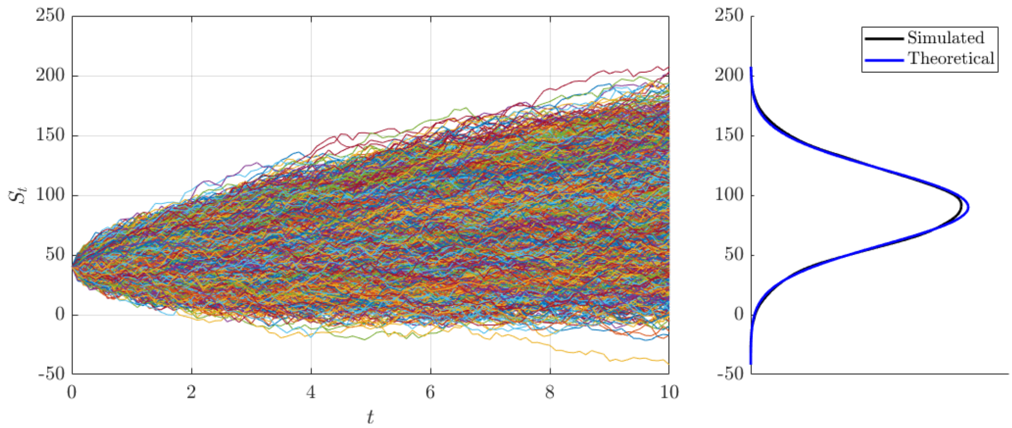

Figure 3.1: Sample paths of $N = 10,000$ Monte Carlo simulations for $\mu = 5$, $\sigma = 10$ and $S_0 = 40$.

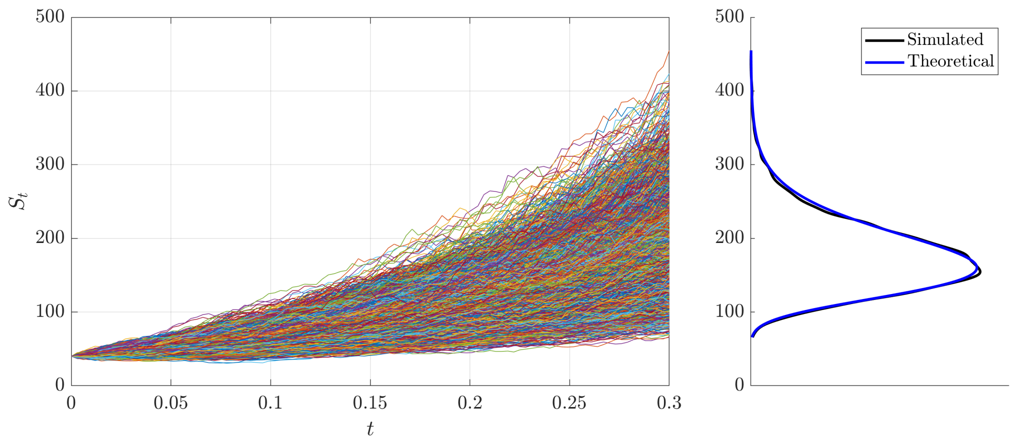

Figure 3.2: Sample paths of $N = 10,000$ Monte Carlo simulations for $\mu = 5S_t$, $\sigma = S_t/2$ and $S_0 = 40$.

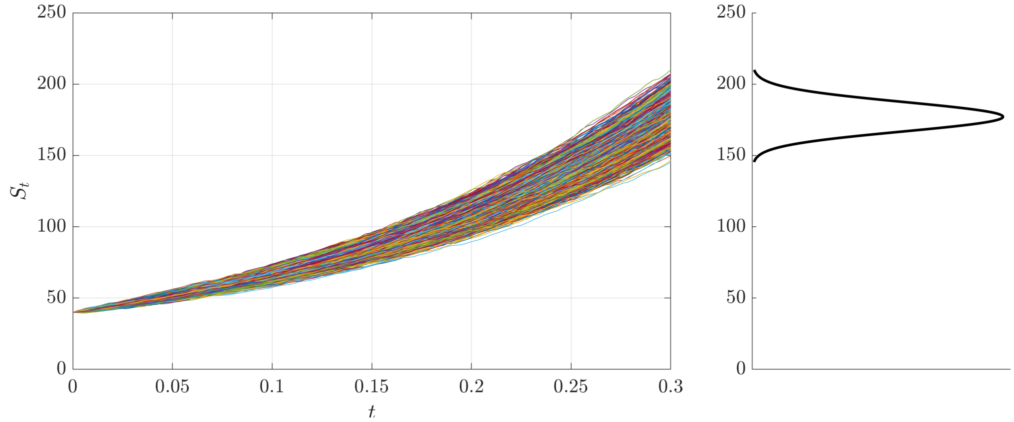

Figure 3.3: Sample paths of $N = 10,000$ Monte Carlo simulations for $\mu = 5S_t$, $\sigma = S_t/2$ and $S_0 = 40$ and $\gamma = 0.6$.

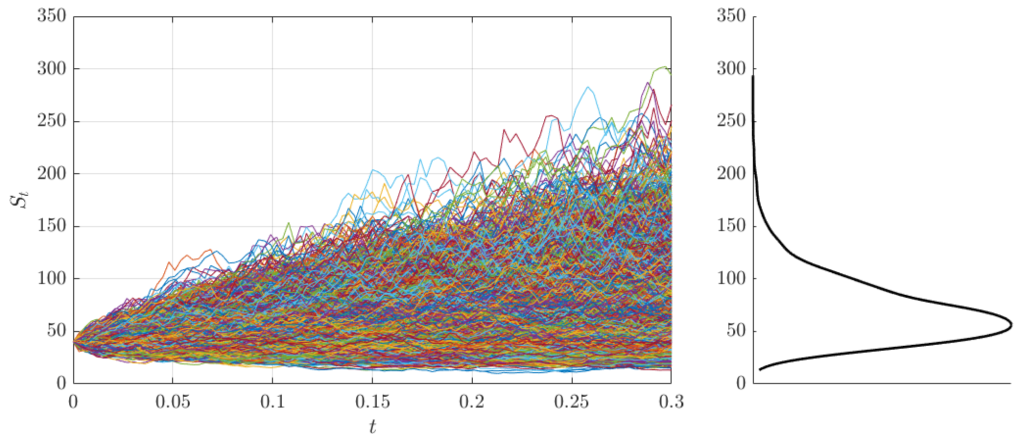

Figure 3.4: Sample paths of $N = 10,000$ Monte Carlo simulations for $\mu = 2(5 - \ln S_t)S_t$, $\sigma = S_t$ and $S_0 = 40$.

3.3 Optimization of the strategy

With the MC simulations we can study the pdf of the payoff for the buy and hold strategy as well as for any strategy proposed. For example, for a linear market we obtain the results obtained in Fig. 3.5 for both the buy and hold and the so-defined strategy 1 for different values of the execution price.

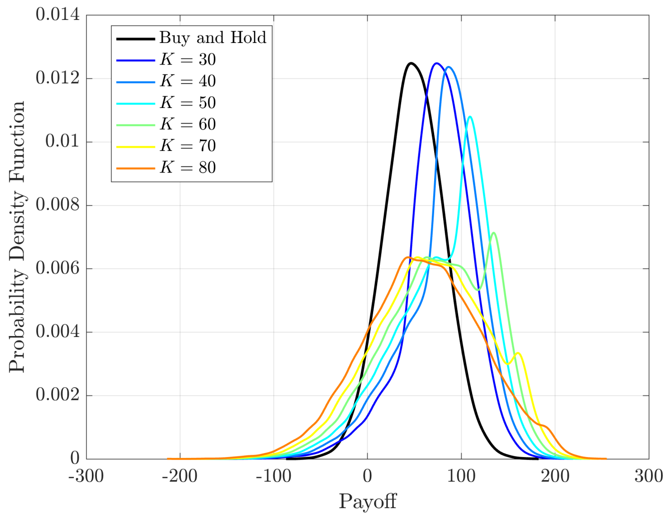

Figure 3.5: Probability density function of the buy and hold strategy and the strategy 1 for different execution prices and the following market parameters: $\mu = 5$, $\sigma = 10$, $S_0 = 40$ and $r = 0.02$.

These results already point out that by using option contracts we are able to modify the payoff distribution. In this case, it strongly depends on the choice of the execution price K. When this starts to get bigger than the actual initial price the tail of the distribution over the losses increases. This means that the risk of losing money is bigger. On the other hand, if the execution price is kept under the initial price, then the distribution is such that increases the revenues without compromising much the losses.

This behavior is clearly suited to be recast as an optimization problem, since there is going to be a perfect choice of parameters that allows the investor to decide between revenues and risk exposure. This is strongly dependent on the risk aversion of the investor and this is out of the scope of this dissertation. Therefore, we are going to adopt a conventional and simplistic, yet widely known and used measure of the performance of each combination of parameters: the Sharpe Ratio (SR). Th SR is defined as [24]

$$SR = \frac{\mathbb{E}[R]}{\sqrt{\mathbb{V}\mathrm{ar}[R]}} , \tag{3.14}$$

where R simply denotes the returns. What this ratio measures is the expected returns and penalizes it with the uncertainty on those returns. Maximizing this ratio implies maximizing the returns while minimizing the variance at the same time. A plot of the SR for different values of the execution price K is displayed in Fig. 3.6.

This simple criterion already yields an optimal value of the execution price, which is $K_{opt} = 11.2$. This fixes all the variables and the investor would be able to make the purchase of the asset and sell the put options.

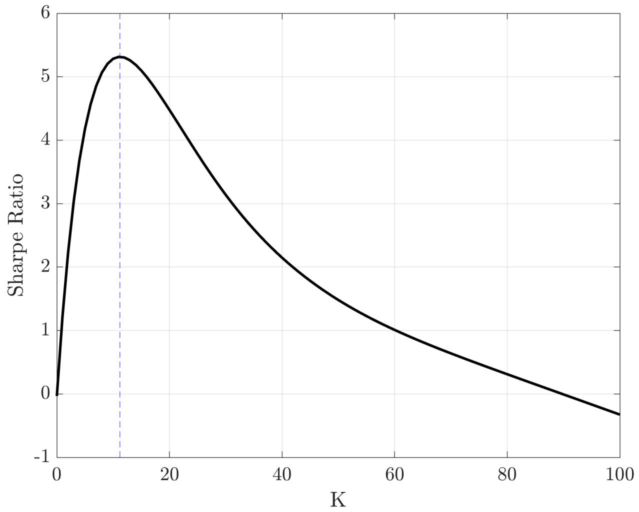

Figure 3.6: Sharpe ratio of the strategy 1 for different execution prices and the following market parameters.

In practice, this is a very simplistic approach and other measures can be used to assess the behavior of a portfolio. For example, the Sortino ratio, defined as [25]

$$SoR = \frac{\mathbb{E}\left[R\right] - R_{\text{target}}}{DR} \,, \tag{3.15}$$

being R_{target} the target expected returns and

$$DR = \sqrt{\int_{-\infty}^{R_{\text{target}}} (R_{\text{target}} - r)^2 p_R(r) \mathrm{d}r} \,, \tag{3.16}$$

where $p_R(r)$ represents the pdf of the returns. This ratio measures the risk-adjusted returns compared to a target investment assuming a downside risk. It is then analogous to Sharpe's ratio but instead of comparing it to a risk-free asset, is compared to this expected or desired portfolio.

Another ratio that might be considered is the so-called upside-potential ratio. It is computed as [26]

$$U = \frac{\mathbb{E}\left[(R_r - R_{\min})_+\right]}{\sqrt{\mathbb{E}\left[(R_r - R_{\min})_-^2\right]}} \,, \tag{3.17}$$

where the new operators are defined as

$$(\xi)_+ = \begin{cases} \xi & \text{if } \xi \geq 0 \\ 0 & \text{else} \end{cases} \,, \tag{3.18}$$

and $(\xi)_= (-\xi)_+$. This ratio measures the upside performance and penalizes the downside risk.

Finally, we can mention the Omega ratio. It is defined as [27]

$$\Omega(\theta) = \frac{\displaystyle\int_{\theta}^{\infty} |1 - F(r)| \mathrm{d}r}{\displaystyle\int_{-\infty}^{\theta} F(r) \mathrm{d}r} \,, \tag{3.19}$$

being $F(r)$ the cumulative distribution function and θ the target return threshold that defines the frontier between gain and loss. This ratio simply measures the probability of winning over the probability of losing. Therefore, it can shed light not about the expected return but rather the chances of improving with respect to the threshold θ.

In practice, since this optimization criteria is strongly dependent on the investor or its client, we will restrict our optimization to that of maximizing the Sharpe ratio. In the end it is a simple, universal, and widely accepted metric and already provides a quantity that measures at the same time the expected returns and the risk in the operation. If in the future another criterion is used, it is left to the reader to check whether the other ratios perform well and really contain a maximum.

4 — Application to a real market

In this last section the methods proposed will be applied in a real financial market. The main idea is to test the range of validity of them and to answer the main question of this dissertation which is whether we can actually improve the revenues by applying option contracts to a value investing scenario.

4.1 Bayesian inference

The first step needed to be able to test our strategies in a real environment is to obtain the parameters of the market. This is, starting from the general stochastic differential equation as

$$\mathrm{d}S_t = \mu(t, S_t|\boldsymbol{\theta})\mathrm{d}t + \sigma(t, S_t|\boldsymbol{\lambda})\mathrm{d}W_t \ , \tag{4.1}$$

where we have explicitly included the dependency of the drift and volatility on two vectors, $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, we will estimate precisely the values of these hyperparameters of each model.

The basic theory of Bayesian Inference (BI) relies on Bayes' Theorem, since the posterior probability of the parameters that we are looking for will be

$$p(\boldsymbol{\theta}, \boldsymbol{\lambda}|\mathbf{S}) = \frac{\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\lambda}|\mathbf{S})\pi(\boldsymbol{\theta}, \boldsymbol{\lambda})}{\mathcal{Z}} \ , \tag{4.2}$$

being $\pi(\boldsymbol{\theta}, \boldsymbol{\lambda})$ the prior distribution of the parameters, $\boldsymbol{S} = \{S_0, S_1, ...S_N\}$ the vector containing the discrete samples at times $t = i\Delta t, \ i \in [0, N]$, the Bayesian evidence defined as

$$\mathcal{Z} = \int p(\boldsymbol{\theta}, \boldsymbol{\lambda}|\alpha)p(\alpha|\mathbf{S})\mathrm{d}\alpha = \int_{\Omega_{\boldsymbol{\Theta}}} \mathcal{L}(\boldsymbol{\Theta})\pi(\boldsymbol{\Theta})\mathrm{d}\boldsymbol{\Theta} \ , \tag{4.3}$$

and the likelihood function [12], [13]

$$\mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\lambda}|\mathbf{S}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2(S_{i-1}|\boldsymbol{\lambda})\Delta t}} \exp\left\{-\frac{1}{2\sigma^2(S_{i-1}|\boldsymbol{\lambda})\Delta t}\left(S_i - S_{i-1} - \mu(S_{i-1}|\boldsymbol{\theta})\Delta t\right)^2\right\} \ . \tag{4.4}$$

To easily deal with the large values that usually the likelihood can take we will use the log-likelihood instead. This is

$$\log \mathcal{L}(\boldsymbol{\theta}, \boldsymbol{\lambda}|\mathbf{S}) = -N\log(\sqrt{2\pi\Delta t}) - \sum_{i=1}^{N} \log\left[\sigma(S_{i-1}|\boldsymbol{\lambda})\right] - \frac{1}{2\Delta t}\sum_{i=1}^{N}\left[\frac{S_i - S_{i-1} - \mu(S_{i-1}|\boldsymbol{\theta})\Delta t}{\sigma(S_{i-1}|\boldsymbol{\lambda})}\right]^2 \ . \tag{4.5}$$

The technique to sample from the posterior distribution that will be used is that known as Nested Sampling (NS). This is a set of algorithms that allow for a fast sampling of the posterior distribution only by evaluating the likelihood and drawing samples of the prior. At the same time it outputs the Bayesian evidence.

Firstly introduced by Ref. [14], this method consists in evaluating the evidence over the prior volume as

$$\mathcal{Z} = \int_0^1 \mathcal{L}(X)\mathrm{d}X \ , \tag{4.6}$$

where

$$X(\lambda) = \int_{\Omega_{\boldsymbol{\Theta}}:\mathcal{L}(\boldsymbol{\Theta})\geq\lambda} \pi(\boldsymbol{\Theta})\mathrm{d}\boldsymbol{\Theta} \ . \tag{4.7}$$

The term X is known as the cumulant prior mass and covers all likelihood values greater than λ. This means that we are dividing the space into different iso-likelihood contours in a high-dimensional space of values defined by λ [15]. Dividing the space by m points ordered such as

$$0 < X_m < ... < X_2 < X_1 < 1$$

allows for a simple estimation of lower bound of the evidence as [14]

$$\mathcal{Z}_{\mathrm{lb}} = \sum_{i=1}^{m}(X_i - X_{i+1})\mathcal{L}_i \tag{4.8}$$

while an upper bound is

$$\mathcal{Z}_{\mathrm{ub}} = \sum_{i=1}^{m}(X_{i-1} - X_i)\mathcal{L}_i + X_m\mathcal{L}_{\mathrm{max}} \ . \tag{4.9}$$

With this information a simple trapezoidal rule or any other traditional integration technique can be used to find the evidence. For example, applying a second-order trapezoidal rule we get that [15]

$$\hat{\mathcal{Z}} = \sum_{\forall i}\hat{p}_i \ , \tag{4.10}$$

with

$$\hat{p}_i = \frac{1}{2}\left[\mathcal{L}(\boldsymbol{\Theta}_{i-1}) + \mathcal{L}(\boldsymbol{\Theta}_i)\right]\left[X_{i-1} - X_i\right] \ . \tag{4.11}$$

The interesting property of this method is that as a consequence we can obtain very easily samples of the posterior probability function as [15]

$$p(\boldsymbol{\theta}, \boldsymbol{\lambda}|\mathbf{S}) = \hat{\mathcal{Z}}^{-1}\sum_{\forall i}\hat{p}_i(\boldsymbol{\Theta}_i)\delta(\boldsymbol{\Theta}|\boldsymbol{\Theta}_i) \ . \tag{4.12}$$

Additionally, from the evidence outputted by the NS algorithm, $\hat{\mathcal{Z}}$, we can easily compute an estimation of the Bayes Factor between two models as

$$\hat{\mathcal{B}} = \frac{\hat{\mathcal{Z}}_{M1}}{\hat{\mathcal{Z}}_{M2}}\frac{\pi(M_1)}{\pi(M_2)} \ , \tag{4.13}$$

and assuming that the prior probabilities of each model is the same and taking the logarithm, the log-Bayes factor is

$$\log\hat{\mathcal{B}} = \log\hat{\mathcal{Z}}_{M1} - \log\hat{\mathcal{Z}}_{M2} \ . \tag{4.14}$$

Jeffrey's criterion states that if $\log\hat{\mathcal{B}} > 2$ there is strong evidence of M_1 being better than model M_2 to explain the data, while if $\log\hat{\mathcal{B}} < 0$ the opposite is true [28].

4.2 S&P500

We can try to apply the strategies employed to one of the most widely known indices: the S&P500. This is one of the most capitalized indices used in the financial market and represents of the order of an 80% of the entire market capitalization [5]. This ETF is constructed taken the 500 most important companies as estimated by the firm Standard & Poors. By its construction, it is one of the most representative indices of the entire situation of the market and, therefore, it is considered as the most stable financial product based on equities [5]. This is why it is interesting to use this index instead of a precise company, but the methodology described is general enough to be adapted to any financial product as long as it admits option trading.

The priors chosen for the different market models employed in this work are detailed in Tab. 4.1 and with all this information the NS can be performed for each one of them.

Model	**Variable**	**Distribution**
Linear	μ	Uniform$(0.1, 5)$
	σ	Uniform$(10, 40)$
Geometric	μ	Uniform$(10^{-4}, 10^{-3})$
	σ	Uniform$(10^{-3}, 10^{-2})$
CEV	μ	Uniform$(10^{-4}, 10^{-3})$
	σ	Uniform$(10^{-2}, 10^{-1})$
	γ	Uniform$(0, 3)$
Schwartz	μ	Uniform$(1, 10)$
	σ	Uniform$(10^{-4}, 10^{-2})$
	γ	Uniform$(10^{-4}, 10^{-2})$

Table 4.1: Priors used for the nested sampling when applied to the S&P500 market for various market models.

In Figs. 4.1, 4.2, 4.3, 4.4 the posterior distributions of the parameters obtained with the nested sampling are presented. In Ref. [12] the Maximum Likelihood Estimators (MLE) are found for the linear model and equal

$$\hat{\mu} = \frac{S_T - S_0}{T} \approx 0.76 \ , \tag{4.15}$$

for the mean and

$$\hat{\sigma} = \sqrt{\frac{1}{T}\sum_{i=1}^{n}(\Delta S_i)^2 - \frac{2}{nT}(S_T - S_0)^2} \approx 15.54 \ , \tag{4.16}$$

for the volatility. These numbers are in extreme agreement with those obtained with the nested sampling algorithm as displayed in Fig. 4.1. Therefore, we can assume that they are valid and use those estimations of the mean and variance to proceed with the optimization of the strategies.

The results of the NS are summarized in Tab. 4.2 while the Bayes factors compared to the linear case are then:

- $\log \mathcal{B}_{\text{Geometric}} = 6,034$
- $\log \mathcal{B}_{\text{CEV}} = 6,033$

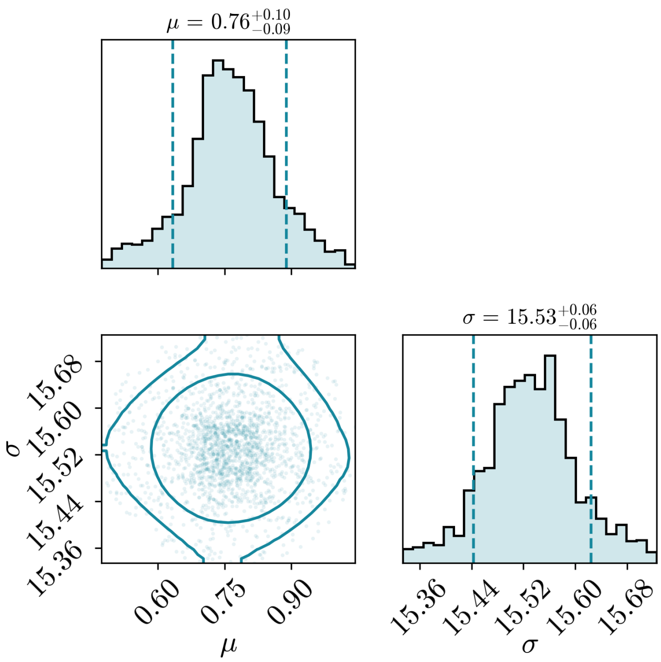

Figure 4.1: Posterior distribution of the parameters for the linear model obtained with the nested sampling algorithm.

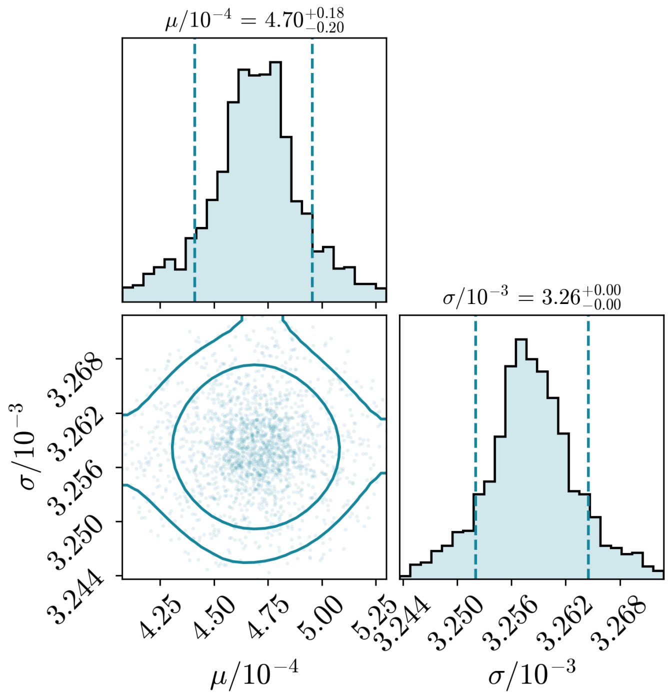

Figure 4.2: Posterior distribution of the parameters for the geometric model obtained with the nested sampling algorithm.

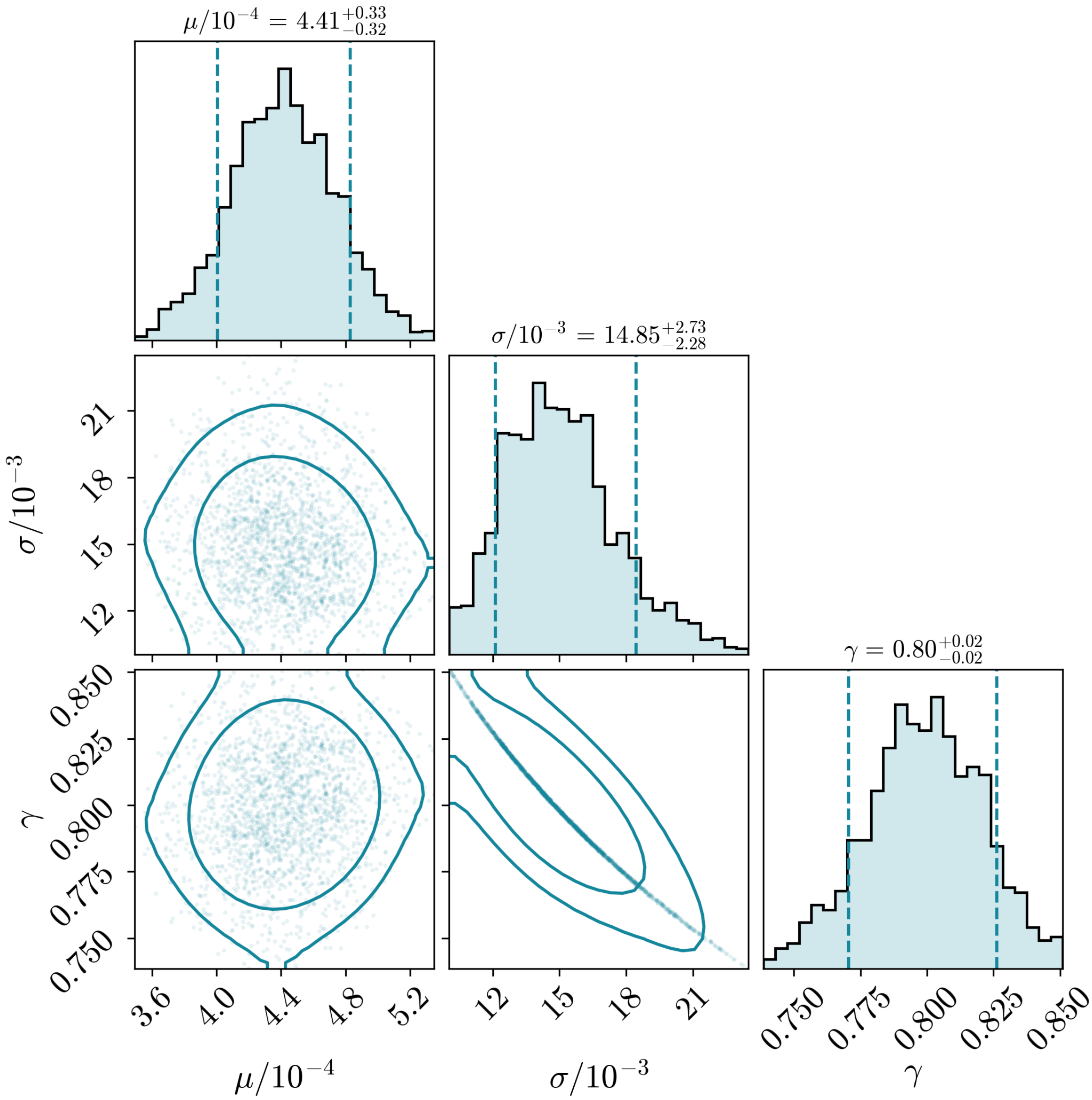

Figure 4.3: Posterior distribution of the parameters for the CEV model obtained with the nested sampling algorithm.

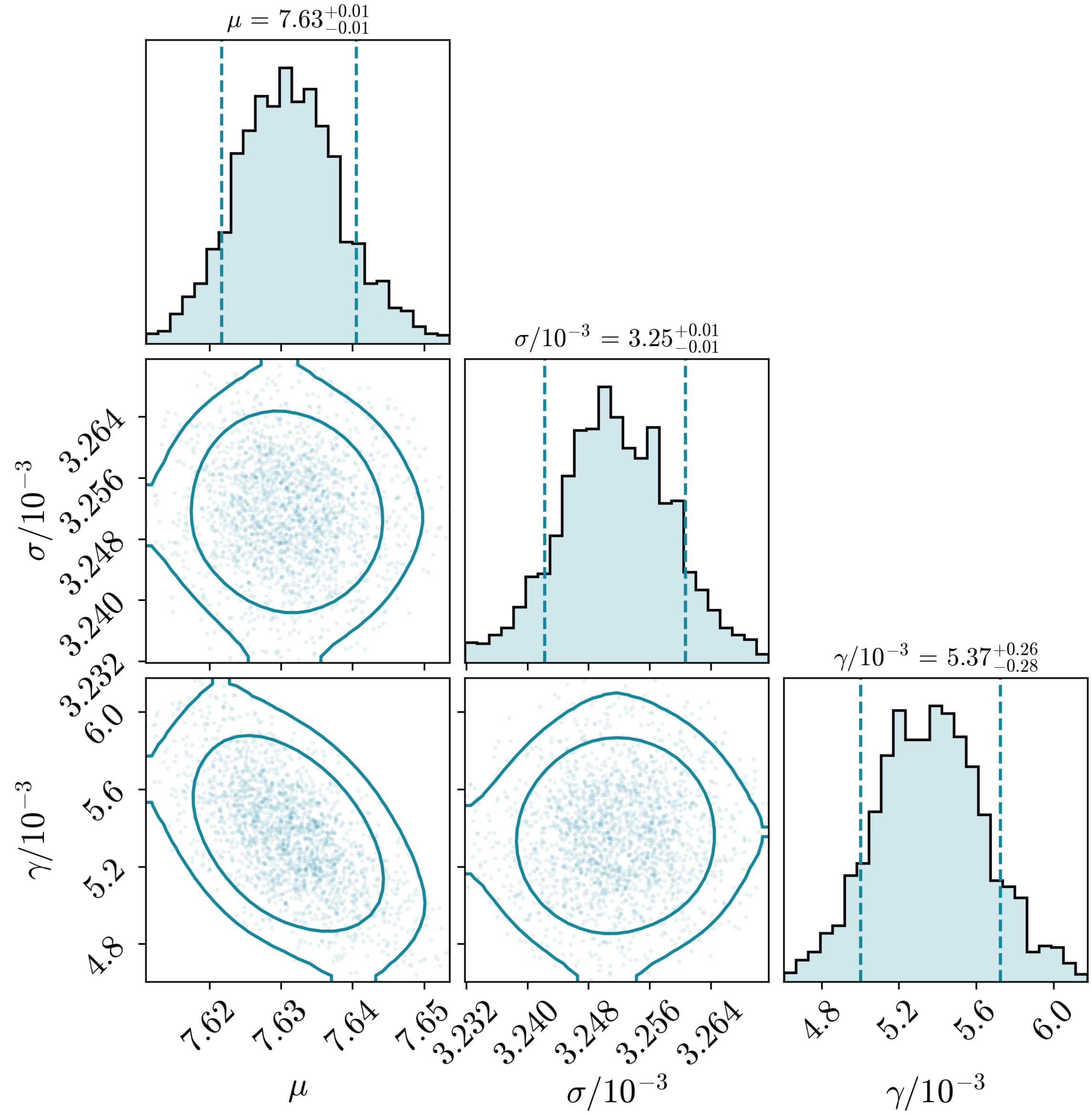

Figure 4.4: Posterior distribution of the parameters for the Schwartz model obtained with the nested sampling algorithm.

- $\log \mathcal{B}_{\text{Schwartz}} = 6,021$

Using Jeffrey's criterion, we can conclude that there is a very strong evidence towards the linear model. We therefore might adopt that one for the rest of the analysis of the real market.

Model	$\log \mathcal{Z}$	μ	σ	γ
Linear	$-4,167$	$0.76^{+0.10}_{-0.09}$	$15.53^{+0.06}_{-0.06}$	–
Geometric	$-10,201$	$4.70\times10^{-4}{}^{+0.18\times10^{-4}}_{-0.20\times10^{-4}}$	$3.26\times10^{-3}{}^{+0.2\times10^{-5}}_{-0.2\times10^{-5}}$	–
CEV	$-10,200$	$4.41\times10^{-4}{}^{+0.33\times10^{-4}}_{-0.32\times10^{-4}}$	$14.85\times10^{-3}{}^{+2.73\times10^{-4}}_{-2.28\times10^{-4}}$	$0.80^{+0.02}_{-0.02}$
Schwartz	$-10,188$	$7.63^{+0.01}_{-0.01}$	$3.25\times10^{-3}{}^{+0.1\times10^{-5}}_{-0.1\times10^{-5}}$	$5.37\times10^{-3}{}^{+0.26\times10^{-3}}_{-0.28\times10^{-3}}$

Table 4.2: Results obtained with the nested sampling when applied to the S&P500 market for various market models.

4.3 Optimizing the parameters of the strategy

With the parameters found in the previous section, we can optimize our strategies' hyperparameters so that are the theoretical optimal ones.

We perform a set of MC simulations using the procedure described in Sec 3.1 with the best fit parameters. The results, alongside the true market evolution is presented in Fig. 4.5. With this MC sample paths we can repeat the procedure of testing the strategy proposed.

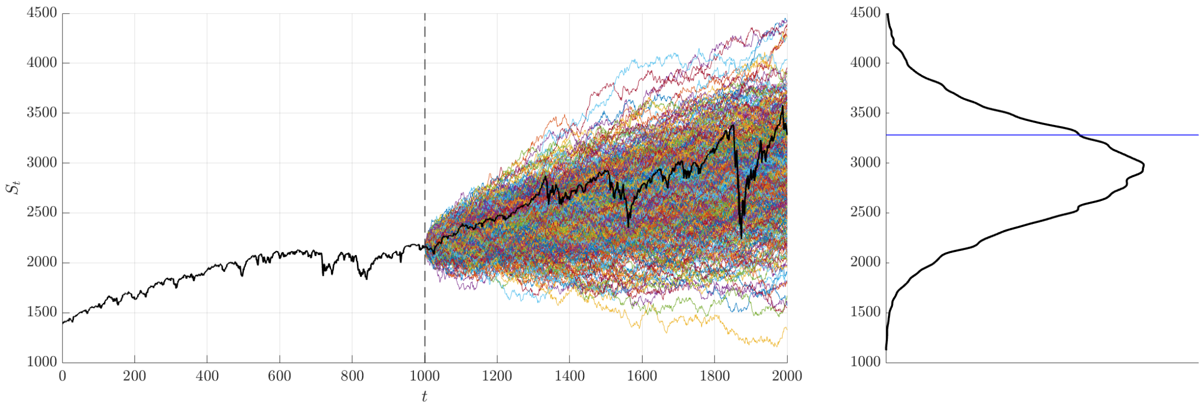

Figure 4.5: Simulated paths for the S&P500 alongside the real data. Time is in trading days after the 21/11/2012. The vertical dashed line indicates the split between the set of data used to find the model hyperparameters and the testing set of data.

The first thing to compute is the pdf of the buy and hold strategy and compare it to that of the strategy 1. This is displayed in Fig. 4.6 for two different values of the risk-free interest rate.

The first interesting conclusion that we can extract is that the level of the risk-free interest rate, which usually is bounded to the bonds, is a key parameter in order for the strategy to succeed. If this is too high, then the premium paid for the puts is so low that the actual strategy yields the same revenue as the buy and hold strategy or does even worse if the execution price is too high. On the other hand, if the risk-free is relatively low (as we have also been seeing after the great recession) then there is margin for this strategy to increase the revenues. It is important to stress out that the way in which the value of the premium has been computed

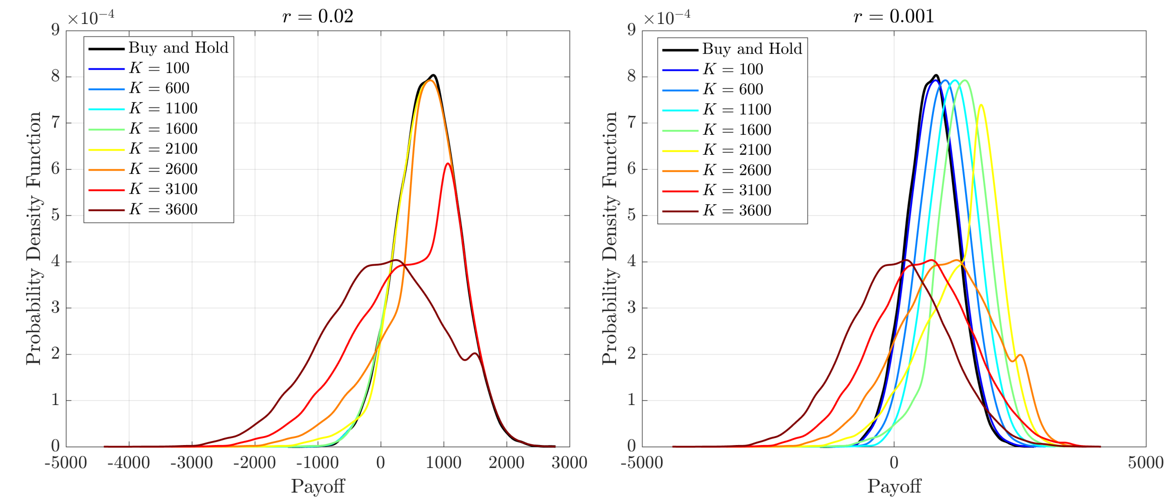

Figure 4.6: Probability density functions for the paths simulated using the best fit parameters for the buy and hold and the strategy 1.

is by applying the Black-Scholes formalism and the real market makers might opt for different calculation techniques and, therefore, yielding different results.

In any case, we can plot the SR for both of these cases and optimize the strategy accordingly. This is displayed in Fig. 4.7.

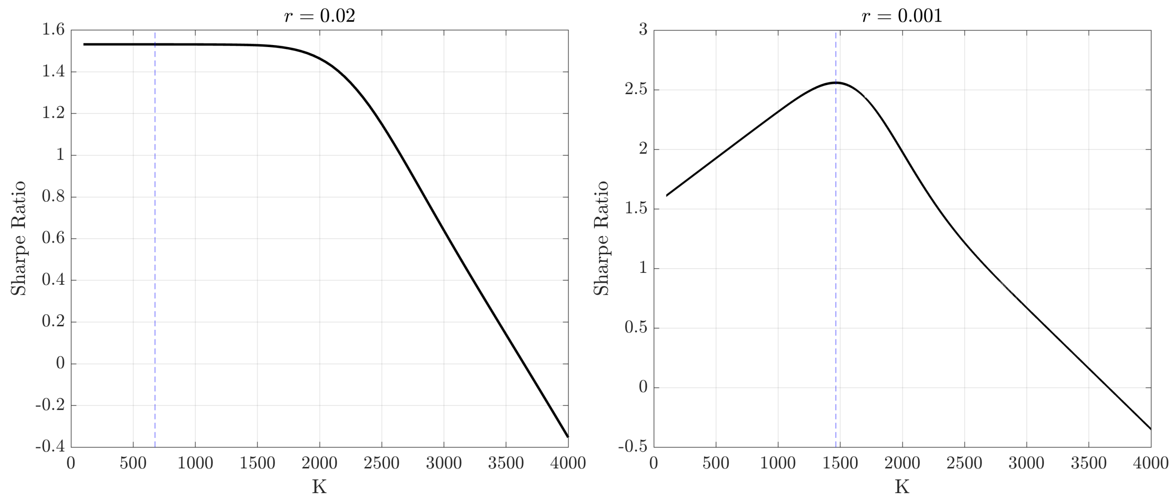

Figure 4.7: Sharpe ratio for different calues of the execution price obtained with the simulated paths of the S&P500 fit parameters.

The optimal execution price for the scenario in which the risk-free rate is of $r = 0.001$ equals $K_{opt} = 1465$. This would be all needed to place the orders in the market.

4.4 Testing the strategies with the real market

Now that the strategies have been tuned to be theoretically optimal in our parametrization of the market, we can finally ask ourselves what would have been the actual outcome if we would have applied it.

The results are summarized in Tab 4.3. It becomes clear how in the context studied we can indeed increase the revenues by using a strategy based on option pricing. The increase in the revenues, though, is highly dependent on the premium paid for the puts and this, in its turn, depends on the market conditions, execution price and maturity. We can only choose the two latter but as displayed in Fig. 4.6 if the market conditions are not favorable there is no choice of a set of parameters that can improve the outcome.

	Payoff	Returns	Annualized returns	Improvement
Buy and Hold	1,129.93	52.5%	19.2%	–
Strategy 1 ($r = 0.02$)	1,129.93	52.5%	19.2%	0%
Strategy 1 ($r = 0.001$)	1,696.51	78.9%	28.8%	9.6%

Table 4.3: Comparison of the performance that would have been obtained with the different strategies if they would have been applied in the S&P500 index.

These results are very promising. In the context of low interest rates, the premiums payed for the puts are high and, therefore, it is a good scenario to apply this strategy. In the particular case of the S&P500 we could have increased the revenues by a 26% by applying this strategy instead of the buy and hold. This is equivalent to an extra 9.6% annually.

As a final remark, it is important to point out that those numbers are subject to the actual conditions that we have established. These are that the pricing of the put is computed using a Black-Scholes model and that we knew the intrinsic value. Both of them can have a strong impact if in a real situation are different or just unknown. Specially, not knowing how a premium is computed can compromise the optimization procedure and, as we have also shown, if this is not correctly chosen it can even increase the losses. Leaving these caveats aside, we have shown that it's still possible to find a situation in which option contracts can really increase the outcome of an investment.

Conclusions

The main question to be answered in this dissertation was whether there is a way to increase the revenues using option contracts under the traditional assumptions made in value investing philosophies. To do so, the mathematical concepts around stochastic differential equations and its analytical and numerical treatment have been explained. To perform the latter, a Monte Carlo approach has been followed and with its results a proper study of the performance of the strategies proposed has been made.

Some of the key results obtained with the mock market are that there are combinations of parameters that allow an increase of the revenues compared to only following a buy and hold strategy. They also provide an evidence of some combinations that are worse. Therefore, there is margin for an optimization of the strategy hyperparameters to find the optimal value that fits the investor's investing style.

Then, a Bayesian inference has been performed over real data of the S&P500 index to find the best stochastic model and its parameters. With this, a set of Monte Carlo simulations has been performed in order to evaluate the strategies performance under this realistic market conditions. The results have pointed out that that the effect of the risk-free interest rate is very relevant. If this is too large, the premium collected of the puts is very small and there is no difference when compared to the buy and hold strategy. On the other hand, when the risk-free interest rate is low enough, then there is room to increase the revenues.

Summarizing, we have shown that indeed it is possible to increase the revenues by using option contracts under the conditions assumed by the value investing community.

Assessment

First of all, I would like to thank the director of this thesis, Prof. Dr. Aslanidis, to support it from the very beginning and for giving me the freedom to develop this project. I know that this topic is not common at all in a Bachelor's degree of this kind. For me it has been very fruitful having done it, since for personal interests I wanted to develop a project more focused on the theoretical side and in a quantitative field. Similarly, it has a real world application in the financial sector and for future job or graduate schools applications it might be an interesting way of showing my interests and capabilities to the recruiters.

Completing this project has been definitely challenging. I have done the vast majority of it while being abroad, as my primary occupation is as a PhD student in Physics and I was visiting the California Institute of Technology (Caltech), in the US, during this semester. Therefore, sometimes meeting the time constraints has been somehow difficult, but in the end, managing to finish it has been a great experience. Also, being able to apply my knowledge of data science and mathematics, which I do as part of my PhD, in the area of economics, has been extremely interesting.

Self-evaluation

This project has many positive aspects that I would like to highlight. The first of them is that it requires more knowledge than the one expected from someone graduating from this Bachelor's degree. The level of mathematics, data science and statistics involved exceeds the contents covered during the studies. Similarly, the methodology employed I believe is new and innovative, as I have not seen it in the available literature. I therefore believe that is a good point for this project to having contributed in advancing the knowledge on the topic with a quantitative approach. A third major point is that of the results. In my opinion they are very good, promising, they correctly answer the question asked at the start of the project and, moreover, they point out that these kind of strategies can produce extra revenues. But this is no longer an intuitive result but rather the result of a thorough study that combines both analytical and numerical techniques that quantitatively support these claims.

On the other hand, there are some caveats or possible improvements that I believe are worth mentioning as well. The first of them is that due to time constraints the amount of strategies tested is not long enough. In the end, numerically only one has been evaluated and put into test and it is, in fact, a very simple one. Similarly, the amount of assumptions carried out, such as the valuation of option premiums using the plain Black-Scholes model or that the intrinsic value is known; may prevent this work from being already prepared to be applied in a real company.

These two major negative points have a straight-forward solution, which is assessing other strategies that might be more complicated and trying to reduce the number of assumptions. Both of them should be considered for future work before implementing them in a real situation with real money at stake. In any case, the methodology employed, the codes used and the theory described in the project establish correctly the grounds to include these changes immediately. In other words, this project has settled the grounds of this analysis.

References

[1] A. Damodaran, *Investment Philosophies: Successful Strategies and the Investors Who Made Them Work* (Finance & Investments). Wiley, 2003, ISBN: 9780471345039.

[2] A. Damodaran, *Investment valuation: Tools and techniques for determining the value of any asset*. John Wiley & Sons, 2012.

[3] J. Greenblatt and A. Tobias, *The Little Book That Still Beats the Market* (Little Books. Big Profits). Wiley, 2010, ISBN: 9780470926710.

[4] R. A. Brealey, S. C. Myers, F. Allen, and P. Mohanty, *Principles of corporate finance*. Tata McGraw-Hill Education, 2012.

[5] S&P Dow Jones Indices, *S&P 500*, [Accessed 17-Dec-2022]. [Online]. Available: `https://www.spglobal.com/spdji/es/indices/equity/sp-500/#overview`.

[6] B. Oksendal, *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[7] L. C. Evans, *An introduction to stochastic differential equations*. American Mathematical Soc., 2012, vol. 82.

[8] S. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models* (Springer Finance Textbooks v. 11). Springer, 2004, ISBN: 9780387401010.

[9] F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *The Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

[10] R. C. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 125–144, Jan. 1976. DOI: `10.1016/0304-405X(76)90022-2`.

[11] S. M. Iacus, *Simulation and inference for stochastic differential equations*. Springer, 2008, vol. 486.

[12] O. Papaspiliopoulos, *An introduction to modelling and likelihood inference with stochastic differential equations*, Lecture Notes from the University of Warwick, 2007.

[13] N. G. Polson and G. O. Roberts, "Bayes factors for discrete observations from diffusion processes," *Biometrika*, vol. 81, no. 1, pp. 11–26, Mar. 1994, ISSN: 0006-3444. DOI: `10.1093/biomet/81.1.11`.

[14] J. Skilling, "Nested sampling for general bayesian computation," *Bayesian analysis*, vol. 1, no. 4, pp. 833–859, 2006.

[15] J. S. Speagle, "Dynesty: A dynamic nested sampling package for estimating bayesian posteriors and evidences," *Monthly Notices of the Royal Astronomical Society*, vol. 493, no. 3, pp. 3132–3158, 2020.

[16] W. R. Inc., *Mathematica, Version 13.1*, Champaign, IL, 2022. [Online]. Available: `https://www.wolfram.com/mathematica`.

[17] MATLAB, *version 7.10.0 (R2022a)*. Natick, Massachusetts: The MathWorks Inc., 2022.

[18] A. Papanicolaou, "Introduction to stochastic differential equations (sdes) for finance," *arXiv preprint arXiv:1504.05309*, 2015.

[19] S. N. Neftci, *Principles of financial engineering*. Academic Press, 2009, ISBN: 9780123735744.

[20] E. S. Schwartz, "The stochastic behavior of commodity prices: Implications for valuation and hedging," *The Journal of finance*, vol. 52, no. 3, pp. 923–973, 1997.

[21] F. Benth, J. Benth, and S. Koekebakker, *Stochastic Modelling of Electricity and Related Markets* (Advanced series on statistical science & applied probability). World Scientific, 2008, ISBN: 9789812812315.

[22] A. W. Bowman and A. Azzalini, *Applied smoothing techniques for data analysis: the kernel approach with S-Plus illustrations*. OUP Oxford, 1997, vol. 18.

[23] B. W. Silverman, *Density estimation for statistics and data analysis*. Routledge, 2018.

[24] W. F. Sharpe, "The sharpe ratio," *Streetwise–the Best of the Journal of Portfolio Management*, pp. 169–185, 1998.

[25] F. A. Sortino and L. N. Price, "Performance measurement in a downside risk framework," *the Journal of Investing*, vol. 3, no. 3, pp. 59–64, 1994.

[26] L. Chen, S. He, and S. Zhang, "When all risk-adjusted performance measures are the same: In praise of the sharpe ratio," *Quantitative Finance*, vol. 11, no. 10, pp. 1439–1447, 2011.

[27] C. Keating and W. F. Shadwick, "A universal performance measure," *Journal of performance measurement*, vol. 6, no. 3, pp. 59–84, 2002.

[28] H. Jeffreys, *The theory of probability*. OUP Oxford, 1998.