Perturbative Corrections for the Anharmonic Oscillator

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This work presents a detailed analysis of the anharmonic oscillator with Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\sqrt{2}\,\hbar\omega\,\frac{\hat{x}^3}{d^3},$$

where $d^2 = \frac{\hbar}{m\omega}$. The $\hat{x}^3$ term is treated as a perturbation and the ground state energy corrections are studied up to order λ^2.

The anharmonic oscillator is studied by perturbation theory, considering a weak cubic term that alters the original harmonic potential. The aim is to understand how small perturbations affect the energy states in quantum systems.

I. THE PROBLEM AND THE HAMILTONIAN

The Hamiltonian considered is

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\sqrt{2}\,\hbar\omega\,\frac{\hat{x}^3}{d^3} \tag{1}$$

where $d^2 = \frac{\hbar}{m\omega}$. The term $\lambda\sqrt{2}\,\hbar\omega\,\frac{\hat{x}^3}{d^3}$ is considered a perturbation of the harmonic Hamiltonian. Perturbation theory is used to calculate the correction in the ground state energy up to second order in λ.

II. GROUND STATE ENERGY CORRECTION

A. First Order Correction

The first order correction in the energy is

$$E_0^{(1)} = \frac{\lambda\sqrt{2}\hbar\omega}{d^3}\langle 0^{(0)}|\hat{x}^3|0^{(0)}\rangle. \tag{2}$$

However, due to the $x \to -x$ symmetry of the unperturbed wave function $|0^{(0)}\rangle$, it follows that $\langle 0^{(0)}|\hat{x}^3|0^{(0)}\rangle = 0$. Hence, $E_0^{(1)} = 0$.

B. Second Order Correction

The expression for the second order correction is

$$E_0^{(2)} = -\sum_{m\neq 0}\frac{|\langle m^{(0)}|\delta H|0^{(0)}\rangle|^2}{E_m^{(0)} - E_0^{(0)}} \tag{3}$$

where $\delta H = \frac{1}{2}\lambda\hbar\omega(a + a^\dagger)^3$. The operator δH is evaluated on the unperturbed state $|0^{(0)}\rangle$

$$\delta H|0^{(0)}\rangle = \frac{1}{2}\lambda\hbar\omega(3|1^{(0)}\rangle + \sqrt{6}|3^{(0)}\rangle)S \tag{4}$$

The contributions to the second-order energy shift are:

$$E_0^{(2)} = -\frac{|\langle 1^{(0)}|\delta H|0^{(0)}\rangle|^2}{E_1^{(0)} - E_0^{(0)}} - \frac{|\langle 3^{(0)}|\delta H|0^{(0)}\rangle|^2}{E_3^{(0)} - E_0^{(0)}} \tag{5}$$

$$\Delta E = -\frac{11}{4}\lambda^2\hbar\omega \tag{6}$$

III. CORRECTED GROUND STATE

The corrected ground state up to first order in λ is

$$|0\rangle = |0^{(0)}\rangle + |0^{(1)}\rangle + \mathcal{O}(\lambda^2),$$

where the first order term $|0^{(1)}\rangle$ is computed as:

$$|0^{(1)}\rangle = -\sum_{m\neq 0}\frac{\langle m^{(0)}|\delta H|0^{(0)}\rangle}{E_m^{(0)} - E_0^{(0)}}|m^{(0)}\rangle \tag{7}$$

$$= -\frac{\langle 1^{(0)}|\delta H|0^{(0)}\rangle}{E_1^{(0)} - E_0^{(0)}}|1^{(0)}\rangle - \frac{\langle 3^{(0)}|\delta H|0^{(0)}\rangle}{E_3^{(0)} - E_0^{(0)}}|3^{(0)}\rangle \tag{8}$$

$$= -\frac{\lambda}{2}\left(3|1^{(0)}\rangle + \frac{\sqrt{6}}{3}|3^{(0)}\rangle\right). \tag{9}$$

IV. EXPECTED VALUE OF $\hat{x}$

For the expected value of $\hat{x}$ in the corrected state, it is calculated as:

$$\langle 0|\hat{x}|0\rangle = \tfrac{d}{\sqrt{2}}\langle 0|(a + a^\dagger)|0\rangle \tag{10}$$

$$= \tfrac{d}{\sqrt{2}}(\langle 0^{(0)}| + \langle 0^{(1)}|)(a + a^\dagger)(|0^{(0)}\rangle + |0^{(1)}\rangle) + \mathcal{O}(\lambda^2) \tag{11}$$

Obtaining as a result:

$$\langle 0|\hat{x}|0\rangle = -\frac{3}{\sqrt{2}}\lambda d + \mathcal{O}(\lambda^2)$$

V. DISCUSSION OF POTENTIAL

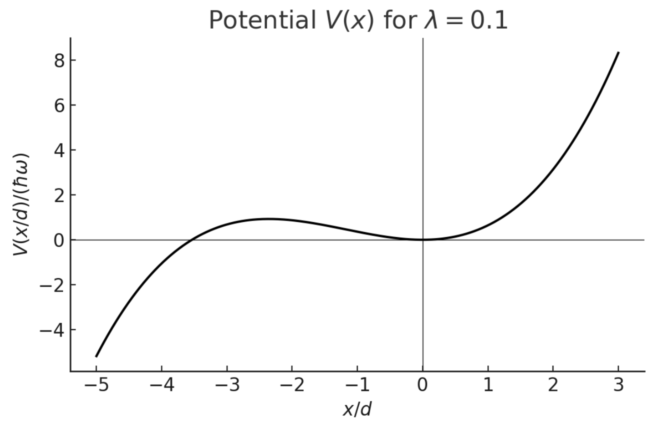

Figure 1. Graph of the potential $V(x)$ as a function of x/d for $\lambda = 0.1$.

The potential is expressed as

$$\frac{1}{\hbar\omega} V(x) = \frac{1}{2}\frac{x^2}{d^2} + \lambda\sqrt{2}\,\frac{x^3}{d^3} \tag{12}$$

For $\lambda > 0$, the cubic term dominates at $x < 0$ and the potential is not bounded below, suggesting the absence of a normalizable ground state.

VI. CONCLUSION

It has been shown how a small cubic perturbation affects the ground state energy and the shape of the potential.